Introduction to Cartan geometry

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February 15, 2022

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I — INTRODUCTION

Cartan geometries are a solution to the very general question: *what is a geometric structure?* Riemannian geometry, conformal geometry and projective geometry are examples of geometric situations.

The mindset is the following. A Cartan geometry should first be a manifold with an homogenous space attached to each point. For instance in Riemannian geometry each point has an attached Euclidean space by equipping the tangent space with the Riemannian metric. This data is then equipped with a Cartan connection explaining how the homogeneous spaces are infinitesimally *connected*.

When one has two different Cartan geometries, one can ask if they are equivalent. For instance, when are two Riemannian manifold isometric or at least locally isometric? This is a deep question known under the general name of the *equivalence problem*. In Riemannian geometry, the differential system $g = \sum \mathrm{d}x_i^2$ asks wether the space is locally euclidean. It is the case if, and only if, a curvature tensor vanishes. Cartan geometries give a similar procedure for all the geometries: a curvature tensor vanishes if, and only if, the space is locally homogeneous.

But when the curvature is not zero, the equivalence problem is harder to solve. What is the meaning of two curvature on two different spaces being equal? Cartan's method for the equivalence problem is a general procedure to study and solve this problem in many situations. An important example is given by the class of the symmetric spaces: those are the Riemannian spaces that are not flat but have a parallel curvature tensor. With Cartan's method one can verify when two spaces with this property are locally equivalent or not.

In this course, we will describe Cartan geometries and introduce the local equivalence problem between geometric structures. The main global problem we will deal with is the classification of smooth Anosov flows on a compact three manifold and, more generally, of non-compact automorphisms groups acting on a compact manifold preserving a contact distribution and two transverse lines contained in the contact plane at each point of the manifold.

II — Lie groups and homogenous spaces

II.1 Lie groups and Lie algebras

We start with the definition of a Lie group. General references for this section are [War83; Kna02].

Definition II.1 *A Lie group is a group G that is also a differential manifold and such that the operations of multiplication and inverse are smooth. That is, the maps $G \times G \to G$ and $G \to G$ given by $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth.*

Definition II.2 *A homomorphism $H \to G$ of Lie groups is a group homomorphism which is a smooth map. The automorphism group of H is the group of bijective homomorphisms of H into H.*

Note that if we ignore continuity in the definition of homomorphisms of Lie groups one might obtain a much larger set.

To each Lie group is associated a Lie algebra which can be thought as the space of tangent vectors at the identity of the group.

Definition II.3 *A Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is a real vector space of finite dimension equipped with a bilinear map*

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{1}$$

satisfying, for any $x, y, z \in \mathfrak{g}$ the anti-commutativity property $[x, y] = -[y, x]$ and the Jacobi identity:

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]. \tag{2}$$

Definition II.4 *A homomorphism $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ between Lie algebras is a homomorphism of vector spaces preserving the Lie bracket, that is, $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{h}$. The automorphism group of $\mathfrak{h}$ is the group of bijective homomorphisms of $\mathfrak{h}$ into $\mathfrak{h}$.*

Let G be a Lie group. If $a \in G$ is fixed, then one can consider the translations $L_a(g) = ag$ and $R_a(g) = ga$ called left and right multiplication respectively.

Definition II.5 *A vector field X on a Lie group G is* left invariant *if, for any $a \in G$, $(L_a)_*(X) = X$. Similarly, it is* right invariant *if $(R_a)_*(X) = X$.*

Note that this condition means $(L_a)_*(X(g)) = X(ag)$.

An important consequence of this definition is that left (or right) invariant vector fields are determined by their value at the identity of the group and the Lie bracket of two invariant vector fields is again invariant. Therefore the set of left invariant vector fields forms a Lie algebra that can be identified to the tangent space of the group at the identity.

Definition II.6 *The Lie algebra of a Lie group G is the set*

$$\mathfrak{g} = \left\{ X \in C^\infty(\mathrm{T}G) \,\middle|\, \forall a \in G,\ (L_a)_*(X) = X \right\} \tag{3}$$

of left invariant vector fields on G equipped with the bilinear map given by the bracket between vector fields.

A subgroup $H \subset G$ which is a Lie group and such that the inclusion map is smooth is a called a Lie subgroup. Imposing that the inclusion is an embedding is equivalent to assuming that the subgroup is closed as a subspace of G (this result is called the closed-subgroup theorem or Cartan theorem).

The relation between Lie algebra homomorphisms and Lie group homomorphisms is described by the following.

Theorem II.7 *Let H and G be Lie groups and $\phi\colon H \to G$ a smooth homomorphism. Then $\mathrm{d}\phi_e\colon \mathfrak{h} \to \mathfrak{g}$ is a homomorphism. Conversely, if $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ is a homomorphism and H is simply connected, then there exists a unique smooth homomorphism $\phi\colon H \to G$ such that $\alpha = \mathrm{d}\phi_e$.*

Examples

(1) The additive group $\mathbf{R}^n$. The automorphism group coincides with linear isomorphisms of $\mathbf{R}^n$, that is to say $\mathrm{GL}(n, \mathbf{R})$. But note that the full group of group automorphisms (not necessarily continuous) of the group $\mathbf{R}^n$ contains non-linear maps.

(2) The set of matrices with determinant one $\mathrm{SL}(n, \mathbf{R})$ and the usual product of matrices as group law.

(3) Let G be a Lie group, $N \subset G$ be a normal subgroup and $K \subset G$ a subgroup satisfying $N \cap K = \{e\}$ and $G = NK$. (This last condition means that $g \in G$ can always be written as nk with $n \in N$ and $k \in K$.) In this conditions, we say that G is the semidirect product of K and N and write $G = N \rtimes K$. Observe that if $g_1 = n_1 k_1$ and $g_2 = n_2 k_2$ then $g_1 g_2 = n_1 (k_1 n_2 k_1^{-1}) k_1 k_2$.

An example is given by the affine linear group $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n \rtimes \mathrm{GL}(n, \mathbf{R})$. Given an affine transformation T acting on the affine plane $\mathbf{R}^n$, the choice of a base point $0 \in \mathbf{R}^n$ allows to write

$$T(x) = c + f(x) \tag{4}$$

with $c \in \mathbf{R}^n$ and $f \in \mathrm{GL}(n, \mathbf{R})$. This decomposition is unique. Hence $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n \,\mathrm{GL}(n, \mathbf{R})$. Note that the change of the base point from $0 \in \mathbf{R}^n$ to $\zeta \in \mathbf{R}^n$ translates to:

$$\zeta + T(x - \zeta) = \zeta + (c - f(\zeta)) + f(x) \tag{5}$$

therefore the linear part f of T is independent from the choice of the base point, but the translational part depends on it.

The composition of two transformations T_1, T_2 is given by:

$$T_1(T_2(x)) = c_1 + f_1(c_2 + f_2(x)) = (c_1 + f_1(c_2)) + f_1 f_2(x) \tag{6}$$

and it proves that $\mathrm{Aff}(\mathbf{R}^n)$ is indeed the semidirect product $\mathbf{R}^n \rtimes \mathrm{GL}(n, \mathbf{R})$.

Note that a convenient representation of the affine group into $\mathrm{GL}(n+1, \mathbf{R})$ is given by

$$(c, f) \mapsto \begin{pmatrix} f & c \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Equivalently, semidirect products $G = N \rtimes K$ are in correspondance with split exact sequences

$$1 \to N \to G \to K \to 1 \tag{8}$$

and in the case of the affine group, we have indeed

$$0 \to \mathbf{R}^n \to \mathrm{Aff}(\mathbf{R}^n) \to \mathrm{GL}(n, \mathbf{R}) \to 1 \tag{9}$$

with the last morphism being independent of the choice of a base point and therefore is indeed restricted to the identity on $\mathrm{GL}(n, \mathbf{R})$.

(4) The three dimensional Heisenberg group Heis(3) is defined as

$$\mathrm{Heis}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| (x, y, z) \in \mathbf{R}^3 \right\} \tag{10}$$

The group law is again the matrix product and is described by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + x' & z + z' + x \cdot y' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Another description of the same group is given by $\mathbf{C} \times \mathbf{R}$ with the (additive) group law

$$(x + \boldsymbol{i} y, z) \cdot (x' + \boldsymbol{i} y', z') = \left((x + x') + \boldsymbol{i}(y + y'), z + z' + \frac{1}{2}(xy' - yx') \right). \tag{12}$$

Both descriptions are compatible. One can start with the Lie algebra:

$$\mathfrak{heis}(3) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{13}$$

The exponential of an element is

$$\exp\left(\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & x & z + \frac{1}{2} xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

Therefore $\exp \colon \mathfrak{heis}(3) \to \mathrm{Heis}(3)$ is a diffeomorphism. The group law furnishes a law on the Lie algebra by taking the logarithm:

$$X \cdot Y = \log(\exp(X) \exp(Y)) = X + Y + \frac{1}{2}[X, Y] \tag{15}$$

and this law on $\mathfrak{heis}(3)$:

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x' & z' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x + x' & z + z' + \frac{1}{2}(xy' - yx') \\ 0 & 0 & y + y' \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

gives the second description.

The automorphism group of a simply connected Lie group coincides with the automorphism group of its Lie algebra. In the case of the Heisenberg group (which is diffeomorphic to $\mathbf{R}^3$) one can use the group operation on the Lie algebra to determine the automorphisms.

Proposition II.8 *The automorphism group of* Heis(3) *(described by coordinates* $(x+\boldsymbol{i}y,t)=(z,t)\in\mathbf{C}\times\mathbf{R}$*) is generated by the following transformations.*

(a) Transformations $(z,t)\mapsto(A(z),t)$ *where* $A\colon\mathbf{C}\to\mathbf{C}$ *is symplectic with respect to the form* $\operatorname{Im}(z\overline{z'})=xy'-yx'$.

(b) Dilations $(z,t)\mapsto(az,a^2t)$*, with* $a\in\mathbf{R}_+^*$.

(c) Conjugations by a translation $(a+\boldsymbol{i}b,c)\in$ Heis(3)*:* $(x+\boldsymbol{i}y,t)\mapsto(x+\boldsymbol{i}y,t+ay-bx)$.

(d) The inversion map $(z,t)\mapsto(\overline{z},-t)$.

Proof We decompose an automorphism $\phi\colon \mathrm{Heis}(3)\to\mathrm{Heis}(3)$ by decomposing its derivative $\mathrm{d}\phi_e\colon \mathfrak{heis}(3)\to\mathfrak{heis}(3)$. With a linear automorphism $\mathrm{d}\phi_e$, we can write $\mathrm{d}\phi_e(x+\boldsymbol{i}y,t)=(A(x,y,t),at+bx+cy)$, with A a linear transformation and a,b,c three real numbers.

We note that an automorphism has to centralize the center of the group: if ζ is in the center, then $0=\mathrm{d}\phi_e[\zeta,\cdot]=[\mathrm{d}\phi_e\zeta,\mathrm{d}\phi_e\cdot]=[\mathrm{d}\phi_e\zeta,\cdot]$. Therefore A can not depend on t. (The center of $\mathfrak{heis}(3)$ is exactly $(0,t)$.)

From $(A(x,y),at+bx+cy)$ one can compose by the conjugation by a translation such that $\mathrm{d}\phi_e$ becomes $(A(x,y),at)$. (Choose the translation $(-c+\boldsymbol{i}b,0)$.)

Next, if a is negative then we compose by an inversion. We obtain $(A'(x,y),|a|t)$ with A' that is either A or $\overline{A}$. Then we can compose by a dilatation by $\lambda=\sqrt{|a|^{-1}}$ so that we obtain $(\lambda A'(x,y),t)$.

Now, because t is fixed, $\lambda A'$ must be a symplectic transformation of $\mathbf{C}$. ▨

Note Hilbert's 5th problem deals with the question of to what extent a topological group has a differential structure. This problem has many interpretations. One of the most important of them was solved by Gleason, Montgomery-Zippin and Yamabe among other contributions: *every connected locally compact topological group without small subgroups (a neighborhood of the identity does not contain a subgroup other than the trivial subgroup) is a Lie group.*

II.1.1 The Maurer-Cartan form

Given a Lie group G and its Lie algebra $\mathfrak{g}$, one might wonder how $\mathfrak{g}$ controls the full tangent space $\mathrm{T}G$. Since G is a group, we can always translate T_eG to any T_gG by doing a left translation L_g or a right translation R_g. We choose to identify any tangent space T_gG with the left translation $(L_g)_*\mathrm{T}_eG$. It implies that $\mathrm{T}G$ has a parallelism $\mathrm{T}G\to G\times\mathfrak{g}$. This parallelism is determined by the Maurer-Cartan form.

Definition II.9 *A manifold* M^n *is parallelisable if there exists* n *vector fields* $(X_1,\dots,X_n)$ *such that at each point* $p\in M$*,* $(X_1(p),\dots,X_n(p))$ *is a basis of* T_pM.

Definition II.10 *The (left) Maurer-Cartan form on a Lie group* G *is the* $\mathfrak{g}$*-valued* 1*-form* θ *defined by*

$$\forall X_g\in\mathrm{T}_gG,\ \theta(X_g)=(L_g)_*^{-1}(X_g)\in\mathfrak{g}. \tag{17}$$

Note Let X be a vector field on G, then $\theta(X)=v$ is constant, if and only if, X is left-invariant and $X(g)=(L_g)_*v$. It furnishes a parallelism of G by choosing a basis of $\mathfrak{g}$.

Recall that for any 1-form α we have

$$\mathrm{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]). \tag{18}$$

Proposition II.11 (Structural equation) *For any $X, Y \in \mathrm{T}_g G$,*

$$\mathrm{d}\theta(X,Y) + [\theta(X), \theta(Y)] = 0. \tag{19}$$

Proof We can evaluate $\mathrm{d}\theta(X,Y)$ by assuming that X, Y are prolongated by left-invariant vector fields X^* and Y^*. For any left-invariant vector field X^*, the image by the Maurer-Cartan form is constant on it by definition. Therefore $X^*(\theta(Y^*))$ and $Y^*(\theta(X^*))$ are both zero. Moreover, since X^*, Y^* are left-invariant, so is $[X^*, Y^*]$ and therefore $\theta([X^*, Y^*]) = [\theta(X), \theta(Y)]$. ▧

Maurer-Cartan form with coordinates The choice of a basis $(e_1, \dots, e_n)$ of $\mathfrak{g}$ allows to write $\theta = (\theta^1, \dots, \theta^n)$ by duality. With X_i the left-invariant vector field verifying $\theta(X_i) = e_i$, we can determine the *structure coefficients*:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k. \tag{20}$$

The structural equation becomes:

$$\mathrm{d}\theta^k(X,Y) = -\sum_{i<j} c_{ij}^k \theta^i \wedge \theta^j. \tag{21}$$

Note Here we use a convention which might be different in some cases (see [KN63] pg. 28) and is sometimes the cause of a factor of $\frac{1}{2}$ in the formula. In fact we define

$$\theta^1 \wedge \theta^2(X,Y) = \theta^1(X) \otimes \theta^2(Y) - \theta^1(Y) \otimes \theta^2(X) \tag{22}$$

in contrast with

$$\theta^1 \wedge \theta^2(X,Y) = \frac{1}{2}\left(\theta^1(X) \otimes \theta^2(Y) - \theta^1(Y) \otimes \theta^2(X)\right). \tag{23}$$

Example Consider the group $\mathrm{SO}(2) \subset \mathrm{GL}(2, \mathbf{R})$. This group is generated by:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{24}$$

By differentiating this parametrization, we obtain

$$\mathrm{d}g_\phi = \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \mathrm{d}\phi \tag{25}$$

Hence, the Lie algebra is also one dimensional and is generated by a single element:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{26}$$

The Maurer-Cartan form translates $\mathrm{d}g_\phi$ for any ϕ to $\mathrm{d}g_0$ by a left translation. Therefore it is given by

$$\theta_\phi = g(\phi)^{-1}\,\mathrm{d}g_\phi \tag{27}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \mathrm{d}\phi \tag{28}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\phi. \tag{29}$$

Matrix groups If $G \subset \mathrm{GL}(n, \mathbf{R})$ is a matrix group with Lie algebra $\mathfrak{g} \subset M_{n\times n}$ one can write the Maurer-Cartan form at $g \in G$ and it is given by $\theta_g = g^{-1}\,\mathrm{d}g$.

Here we interpret $\mathrm{d}g$ as the differential of the embedding of G into the space of matrices $M_{n\times n}$. In coordinates g_{ij} of that embedding, one has $\theta_g = g_{ik}^{-1}\,\mathrm{d}g_{kj}$, which is a $\mathfrak{g}$-valued 1-form.

Vector space valued forms The Maurer-Cartan form is an example of vector space valued form. We define the wedge product of a V_1-valued 1-form θ_1 and a V_2-valued 1-form θ_2 to be the $V_1 \otimes V_2$-valued form

$$\theta_1 \wedge \theta_2(X, Y) = \theta_1(X) \otimes \theta_2(Y) - \theta_1(Y) \otimes \theta_2(X). \tag{30}$$

If there exists a bilinear map $[\cdot,\cdot] : V \times V \to V$ we note the composition of $\wedge$ (for 1-forms) and $[\cdot,\cdot]$ by

$$[\theta_1 \wedge \theta_2](X, Y) := [\theta_1(X), \theta_2(Y)] - [\theta_1(Y), \theta_2(X)]. \tag{31}$$

Observe then that $[\theta_1(X), \theta_2(Y)] = \frac{1}{2}[\theta_1 \wedge \theta_2](X, Y)$.

Writing, in general, θ_n for a $\mathfrak{g}$-valued n-form we may define the exterior derivative and the product of two forms accordingly. We easily verify:

(1) $[\theta_p \wedge \theta_q] = (-1)^{pq}[\theta_q \wedge \theta_p]$,

(2) $(-1)^{pr}[[\theta_p \wedge \theta_q] \wedge \theta_r] + (-1)^{qr}[[\theta_r \wedge \theta_p] \wedge \theta_q] + (-1)^{qp}[[\theta_q \wedge \theta_r] \wedge \theta_p]$.

Moreover,

$$\mathrm{d}[\theta_p \wedge \theta_q] = [\mathrm{d}\theta_p \wedge \theta_q] + (-1)^{pq+1}[\theta_p \wedge \mathrm{d}\theta_q]. \tag{32}$$

Darboux derivatives

A Maurer-Cartan form allows the computation of Darboux derivatives.

Definition II.12 *If $f \colon M \to G$ is smooth and if θ is the Maurer-Cartan form of G then the Darboux derivative of f is:*

$$f^*\theta = \theta f_*. \tag{33}$$

Example 1 In $\mathbf{R}^n$ the Darboux derivative is in a sense closer to the usual derivative than the differential. Indeed, recall that if $f \colon \mathbf{R}^p \to \mathbf{R}^n$ is smooth, then

$$\forall (x, v) \in \mathrm{T}\mathbf{R}^n,\ f_*(x, v) = (f(x), \mathrm{d}f_x(v)). \tag{34}$$

The consideration of f_* or even $\mathrm{d}f$ depends strongly on the consideration of a base point. But with the Darboux derivative, the tangent spaces are connected:

$$f^*\theta(x, v) = \theta(f(x), \mathrm{d}f_x(v)) = [\mathrm{d}f_x(v)] \tag{35}$$

and the class of $[\mathrm{d}f_x(v)]$ belongs to a single copy of $\mathbf{R}^n$.

Example 2 One parameter subgroups of a group G are defined by elements of the Lie algebra. For any $x \in \mathfrak{g}$ one defines a homomorphism

$$\exp_x \colon \mathbf{R} \to G, \tag{36}$$

which is the unique homomorphism satisfying $\exp_x^* \theta = x$.

Definition II.13 *The exponential map* $\exp \colon \mathfrak{g} \to G$ *is defined by*

$$\exp(x) = \exp_x(1). \tag{37}$$

Although exp has several properties analogous to the real exponential, due to the non-commutativity, one has a more complicated formula for the product of two exponentials (it is the Baker-Campbell-Hausdorff formula which is only valid locally):

$$\exp(x)\exp(y) = \exp\left(x + y + \frac{1}{2}[x, y] + \cdots\right). \tag{38}$$

If $\phi \colon H \to G$ is a group homomorphism one has

$$\exp \circ \mathrm{d}\phi_e = \phi \circ \exp_e. \tag{39}$$

Lemma II.14 *Let X be a left-invariant vector field. Then its flow is $R_{\exp(tx)}$ with $x = \theta(X)$.*

PROOF Since X is left-invariant, so must be its flow. Therefore the integral curve at $g \in G$ is given by $L_g \exp(tx) = R_{\exp(tx)} g$. Hence the flow is given by $R_{\exp(tx)}$. ▩

II.1.2 The adjoint representation

An action of a Lie group G on a manifold induces a representation of the group on the automorphism group of the tangent space of a fixed point of the action. For, let $\phi\colon G \times M \to M$ be an action with a fixed point $G \cdot p = p$ at $p \in M$. Then for every $g \in G$, $\mathrm{d}\phi_{(g,p)}$ acts on $\mathrm{T}_p M$ as a linear isomorphism. It furnishes a representation $g \mapsto \mathrm{d}\phi_{(g,p)} \in \mathrm{Aut}(\mathrm{T}_p M)$.

In particular the adjoint action $G \times G \to G$ defined by $(g,h) \mapsto ghg^{-1}$ induces the representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ (observe that $\mathrm{Aut}(\mathrm{T}_e G)$ is isomorphic to $\mathrm{GL}(n, \mathbf{R})$ with $n = \dim_{\mathbf{R}} G$). For $g \in G$, Ad_g is the automorphism

$$\mathrm{Ad}_g(X) = \mathrm{d}(h \mapsto ghg^{-1})_e(X) = (L_g)_*(R_{g^{-1}})_* X \tag{40}$$

The adjoint representation is also exactly what we need to compare the Maurer-Cartan form θ defined by left-invariance with the action by right translations.

PROPOSITION II.15 *For any $g \in G$, the Maurer-Cartan form θ verifies*

$$R_g^* \theta(X) = \mathrm{Ad}_g^{-1}(\theta(X)). \tag{41}$$

PROOF Assume that $X = (L_x)_* v$. By the preceding definition, we have:

$$R_g^* \theta(X) = \theta((R_g)_* X) \tag{42}$$
$$= \theta((R_g)_*(L_x)_* v) \tag{43}$$
$$= \theta((L_x)_*(R_g)_* v) \tag{44}$$
$$= \theta((R_g)_* v) \tag{45}$$
$$= (L_g)_*^{-1}(R_g)_* v = \mathrm{Ad}_g^{-1} v. \tag{46}$$

▩

The differential of Ad_g at the origin $g = e$ is denoted by $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathrm{T}_e G)$:

$$\mathrm{ad}_x = \mathrm{dAd}_e(x). \tag{47}$$

It is in fact given by the bracket of the Lie algebra.

LEMMA II.16 *Let $x, y \in \mathfrak{g} \cong \mathrm{T}_e G$. Then*

$$\mathrm{dAd}_e(x, y) = \mathrm{ad}_x(y) = [x, y]. \tag{48}$$

PROOF Let X, Y be the left-invariant vector fields prolongating x and y. That is to say, $\theta(X) = x$ and $\theta(Y) = y$ with θ the Maurer-Cartan form. First, observe the identity:

$$\mathrm{d}^2\mathrm{id}(V, W) = V(\mathrm{id}(W)) - W(\mathrm{id}(V)) - [V, W] \tag{49}$$

showing that, since $\mathrm{d}^2\mathrm{id} = 0$,

$$[V, W] = V(\mathrm{id}(W)) - W(\mathrm{id}(V)). \tag{50}$$

On the other hand we have

$$\mathrm{dAd}(X, Y) = X(\mathrm{Ad}(Y)) - Y(\mathrm{Ad}(X)) - \mathrm{Ad}([X, Y]) \tag{51}$$

We prove that

$$X(\mathrm{Ad}(Y))|_e = 2X(\mathrm{id}(Y))|_e. \tag{52}$$

Recall that the flow ϕ^t of X is $R_{\exp(tX)}$. Hence on one hand:

$$X(\mathrm{id}(Y))|_e = \lim_{t=0} \frac{(\phi_e^{-t})_* Y(\phi_e^t(e)) - Y(e)}{t} \tag{53}$$
$$= \lim_{t=0} \frac{(R_{\exp(-tX)})_*(L_{\exp(tX)})_* Y(e) - Y(e)}{t} \tag{54}$$
$$= \lim_{t=0} \frac{\mathrm{Ad}(\exp(tX))_* Y(e) - Y(e)}{t} \tag{55}$$

and on the other hand (note that $\mathrm{Ad}_\phi(Y)$ is again left-invariant):

$$X(\mathrm{Ad}(Y))|_e = \lim_{t=0} \frac{(\phi_e^{-t})_* \mathrm{Ad}_{\phi_e^t(e)}(Y)(\phi_e^t(e)) - Y(e)}{t} \tag{56}$$
$$= \lim_{t=0} \frac{(R_{\exp(-tX)})_*(L_{\exp(tX)})_*(L_{\exp(tX)})_*(R_{\exp(-tX)})_* Y(e) - Y(e)}{t} \tag{57}$$
$$= \lim_{t=0} \frac{\mathrm{Ad}(\exp(2tX))_* Y(e) - Y(e)}{t} \tag{58}$$
$$= 2X(\mathrm{id}(Y))|_e. \tag{59}$$

To conclude, we observe that at $e \in G$:

$$\mathrm{dAd}(X,Y)|_e = X(\mathrm{Ad}(Y)) - Y(\mathrm{Ad}(X)) - \mathrm{Ad}([X,Y]) \tag{60}$$
$$= 2X(\mathrm{id}(Y))|_e - 2Y(\mathrm{id}(X))|_e - \mathrm{Ad}_e[X,Y]|_e \tag{61}$$
$$= 2X(\mathrm{id}(Y))|_e - 2Y(\mathrm{id}(X))|_e - [X,Y]|_e \tag{62}$$
$$= [X,Y]|_e. \tag{63}$$

▩

More generally, we have:

Proposition II.17 *The differential of the representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ *at* $g \in G$ *computed at the vector* $X^* = (L_g)_* X \in \mathrm{T}_g G$ *is*

$$\mathrm{dAd}_g(X)(Y) = \mathrm{Ad}_g(\mathrm{ad}_X(Y)). \tag{64}$$

Proof Writing a path through g as $L_g\gamma(t)$ with $\gamma(0) = e$ and $\dot\gamma(0) = X$ we have $\mathrm{Ad}_{L_g\gamma(t)}(Y) = \mathrm{Ad}_g \circ \mathrm{Ad}_{\gamma(t)}(Y)$. Therefore

$$(\mathrm{dAd}_g(X))(Y) = \left.\frac{\mathrm{dAd}_g \circ \mathrm{Ad}_{\gamma(t)}}{\mathrm{d}t}\right|_{t=0}(Y) = \mathrm{Ad}_g \circ \mathrm{ad}_X(Y). \tag{65}$$

▩

The adjoint automorphism by $g \in G$ fits in the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{g(\cdot)g} & G \end{array} \tag{66}$$

and the adjoint representation satisfies

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}) \end{array} \tag{67}$$

II.2 Homogeneous spaces

Homogeneous spaces will be the flat model geometries. They appear naturally when there exists a transitive action. Indeed, if $G \times M \to M$ is a transitive action one can identify M with the quotient ${}^{G}\!/_{H_x}$ where H_x is the isotropy subgroup of a chosen element $x \in M$. A different choice $gx \in M$ gives rise to the isotropy $H_{gx} = gH_xg^{-1}$.

Definition II.18 *A homogeneous space is a differential manifold obtained by the quotient of a Lie group G by a closed Lie subgroup $H \subset G$. We note the set of left cosets gH by ${}^{G}\!/_{H}$.*

The group G acts transitively on the homogeneous space ${}^{G}\!/_{H}$ by left translations, the isotropy subgroup at the identity being H.

Note If H were not closed then the quotient ${}^{G}\!/_{H}$ would not separated with the quotient topology.

Examples

(1) *The Euclidean space.*

The group of the isometries of the Euclidean space is $\mathrm{Eucl} = \mathbf{R}^n \rtimes \mathrm{O}(n)$. It acts on $\mathbf{R}^n$ with isotropy $\mathrm{O}(n)$. Therefore $\mathbf{R}^n = {}^{\mathrm{Eucl}}\!/_{\mathrm{O}(n)}$ as homogeneous space.

(2) *The hyperbolic space.*

Hyperbolic space is the simply connected complete constant negative sectional curvature Riemannian space. Its connected isometry group is $\mathrm{SO}(n,1)$ with isotropy $\mathrm{SO}(n)$. Here $\mathrm{SO}(n,1)$ is the group preserving the quadratic form

$$\begin{pmatrix} \mathrm{id}_{\mathbf{R}^n} & 0 \\ 0 & -1 \end{pmatrix}. \tag{68}$$

(3) *The similarity group acting on $\mathbf{R}^n$.*

The connected similarity group is the group $\mathrm{Sim}(\mathbf{R}^n) = \mathbf{R}^n \rtimes (\mathbf{R}_+^* \times \mathrm{O}(n))$. It is a subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$. Transformations of $\mathbf{R}_+^* \times \mathrm{O}(n)$ are of the form $\lambda P(x)$ with $\lambda > 0$ and P an orthogonal transformation.

The similarity group is the conformal group acting on $\mathbf{R}^n$. (Each conformal transformation has to be defined on the full space $\mathbf{R}^n$.) Therefore, it consists of the transformations of $\mathbf{R}^n$ which preserve angles. The isotropy at the origin is $\mathbf{R}_+^* \times \mathrm{O}(n)$.

(4) *The conformal sphere.*

There are more conformal transformations than just $\mathrm{Sim}(\mathbf{R}^n)$. But those are not defined strictly on $\mathbf{R}^n$ but rather on the one-point compactification S^n. The conformal sphere is the homogeneous space $\mathrm{PO}(n+1,1)/\mathrm{Sim}(\mathbf{R}^n)$.

(5) *The projective space.*

The projective space $\mathbf{RP}^n$ is the homogenous space $\mathrm{GL}(n+1,\mathbf{R})/H$ where

$$H = \left\{ \begin{pmatrix} \star & \star \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{GL}(n,\mathbf{R}) \right\}. \tag{69}$$

(6) *Flag spaces.*

The projective space is an example of flag spaces. A flag is a sequence $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbf{F}^n$ for any field $\mathbf{F}$. For instance, the projective space $\mathbf{FP}^n$ is the set of lines in $\mathbf{F}^{n+1}$.

A complete flag is a flag with $\dim V_i = i$. They are maximal in length. When $\mathbf{F} = \mathbf{C}$ we get an homogeneous space structure with the quotient

$${}^{\mathrm{SU}(n)}\!/_{\mathrm{S}(\mathrm{U}(1) \times \cdots \times \mathrm{U}(1))}. \tag{70}$$

(7) *Stiefel manifolds.*

The space of orthonormal k-frames in $\mathbf{R}^n$ (with $0 < k < n$) is the Stiefel manifold $S(k,n)$. It is possible to show that

$$S(k,n) = {}^{\mathrm{SO}(n)}\!/_{\mathrm{SO}(n-k)}. \tag{71}$$

(8) *Every manifold is a homogeneous space.*

The full group of the diffeomorphisms of a manifold is not a Lie group but might be described by an analogous structure with infinite dimension.

The easiest situation is for a compact manifold, say M. The smooth diffeomorphism group $\mathrm{Diff}^\infty(M)$ has a structure of a Fréchet Lie group which is homeomorphic to the space of smooth vector fields. The group $\mathrm{Diff}^\infty(M)$ acts transitively on M. Therefore, any manifold can be considered as a homogeneous space $\mathrm{Diff}^\infty(M)/H$, where H is the isotropy at a point in M, that is to say, the set of diffeomorphisms fixing the point. We will not deal with infinite dimension Lie groups.

Construction à la Cartan We can reproduce how Cartan described the construction of the Maurer-Cartan form at the early stages of the theory. In fact, we here describe the main technique of the moving frame (*repère mobile*) that Cartan attributes to Darboux.

Consider the affine space $\mathbf{R}^3$. At any point $m \in \mathbf{R}^3$, associate a frame (e_1, e_2, e_3) base at m. The map (e_1, e_2, e_3) should be smooth depending on m.

The infinitesimal change of m by δm can be expressed by:

$$\delta m = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3. \tag{72}$$

It gives a 1-form with values in $\mathbf{R}^3$.

The infinitesimal change of a base vector e_i by δe_i can be described by the image of an infinitesimal matrix acting on (e_1, e_2, e_3):

$$\delta e_i = \omega_i^1 e_1 + \omega_i^2 e_2 + \omega_i^3 e_3 \tag{73}$$

and this furnishes a 1-form with values in $\mathfrak{gl}(3)$.

Those four 1-forms $\theta = (\delta e_1, \delta e_2, \delta e_3, \delta m)$ compose the Maurer-Cartan form of the affine space.

II.2.1 The tangent space

With a homogeneous space G/H the tangent space can be described infinitesimally and the action of G (on the left) can be measured.

At eH, the tangent space is naturally isomorphic to $\mathfrak{g}/\mathfrak{h}$ as linear spaces. Therefore, the tangent bundle of the homogenous spaces $\mathrm{T}G/H$ can be seen as a quotient of the trivial bundle

$$G \times_H \mathfrak{g}/\mathfrak{h}. \tag{74}$$

The quotient will be by the right action of H:

$$(g, v) \cdot h \sim (gh, \mathrm{Ad}(h)^{-1} v). \tag{75}$$

Note that at the isotropy $H \subset G$, the action of $h \in H$ on a point pH is $hpH = hph^{-1}H$ and therefore H acts on $\mathrm{T}_{eH} G/H$ by $\mathrm{Ad}(h)$.

Proposition II.19 *There exists a canonical isomorphism*

$$\mathrm{T}G/H \cong G \times_H \mathfrak{g}/\mathfrak{h}. \tag{76}$$

Proof Let $\pi\colon G \to G/H$ be the quotient map. Let $\phi\colon G \times \mathfrak{g}/\mathfrak{h} \to \mathrm{T}G/H$ be defined by

$$\phi(g, v) = (gH, \pi_*(L_g)_* v). \tag{77}$$

We prove that this map is well defined in the quotient by the right action of H. Note that $\pi_*(R_h)_* = \pi$ since $\pi \circ R_h = \pi$ and $\pi_*(L_g)_* = (L_g)_*\pi_*$.

$$\phi((g,v)\cdot h) = \phi(gh, \mathrm{Ad}(h)^{-1}v) \tag{78}$$

$$= (ghH, \pi_*(L_{gh})_*\mathrm{Ad}(h)^{-1}v) \tag{79}$$

$$= (gH, (L_g)_*\pi_*(R_h)_*v) \tag{80}$$

$$= (gH, (L_g)_*\pi_*v) = \phi(g,v) \tag{81}$$

We can check that this morphism is injective at every point. If $\phi(g,v) = (gH, 0)$ then $\pi_* v = 0$ and therefore $v \in \mathfrak{h}$. It is surjective by dimensionality. ▧

II.2.2 Effective pairs

It is important to keep track of both groups G and H and not only their quotient space. On the other hand it is reasonable to consider only connected quotients G/H.

Definition II.20 *We will refer as a Klein geometry a pair (G,H) such that the homogeneous space G/H is connected.*

There are two conditions which one can add without much loss of generality, namely, that the action of G be effective and that G be connected.

Note that if $g \in G$ acts trivially on G/H then $geH = eH$ and therefore $g \in H$. Let $h \in H$ be acting trivially. For any $g \in G$ and any coset pH we would have that $ghg^{-1}pH = g(h(g^{-1}pH))$ is equal to $g(g^{-1}pH)$ since h acts trivially on $g^{-1}pH$ and therefore $ghg^{-1}pH = pH$. So if h acts trivially, then ghg^{-1} does too.

Definition II.21 *We say that a maximal subgroup $K \subset H$ which is normal in G is the kernel of a Klein geometry. The action of K is trivial and we say that the geometry is effective if $K = \{e\}$.*

If K is the maximal normal subgroup in H (the definition implies that K is a closed subgroup of G) one can consider the effective geometry $(G/K, H/K)$ which describes the same homogeneous space as $(G/K)/(H/K)$. It is diffeomorphic to G/H with an equivariant action by G/K.

Sometimes one might consider non-effective Klein geometries. For instance, $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$ corresponds to the hyperbolic geometry but the subgroup $\mathbf{Z}_2 \subset \mathrm{SL}(2,\mathbf{R})$ generated by $-\mathrm{id}$ is a maximal normal subgroup contained in $\mathrm{SO}(2)$. Nonetheless, this subgroup is discrete and is does not intervene infinitesimally.

If G is not connected one can consider the connected component containing the identity $G_e \subset G$ and we obtain that G/H is diffeomorphic to $G_e/(H \cap G_e)$ with an equivariant action by G_e. This follows since if G/H is connected, one has $G = G_e H$. On the other hand, one can prove that if H is connected then G is also connected.

Lemma II.22 *Let $N \subset G$ be a normal subgroup with corresponding algebras $\mathfrak{n} \subset \mathfrak{g}$. Then for all $v \in \mathfrak{g}$ and $n \in N$,*

$$\mathrm{Ad}_n(v) - v \in \mathfrak{n}. \tag{82}$$

Proof Since N is normal, for any $g \in G$ and any $n \in N$ we have $ngn^{-1}g^{-1} \in N$. Let $g(t) = \exp(tv)$. We have:

$$(L_n L_{g(t)} R_{n^{-1}})g(-t) \in N \tag{83}$$

and by derivation at $t = 0$:

$$\mathrm{Ad}_n(v) - v \in \mathfrak{n}. \tag{84}$$

▧

Reciprocally, this condition implies, by differentiation along a path in N, that $[\mathfrak{n},\mathfrak{g}] \subset \mathfrak{n}$ so $\mathfrak{n}$ is an ideal of G.

We will need to identify maximal normal subgroups of G contained in $H \subset G$. The goal is to obtain properties for effective Klein geometries. The easiest way to start is with a normal subgroup N of H ($N = H$ is the most natural choice) so that its Lie algebra $\mathfrak{n}$ is an ideal of $\mathfrak{h}$. According to the preceding lemma, a candidate for a normal subgroup of G contained in $N \subset H$ is

$$N' = \left\{n \in N \mid \forall\, v \in \mathfrak{g},\ \mathrm{Ad}_n\, v - v \in \mathfrak{n}\right\}. \tag{85}$$

The subgroup N' might be much smaller that N. At least, it is still normal in H:

$$\mathrm{Ad}_{hnh^{-1}}(v) - v = \mathrm{Ad}_h(\mathrm{Ad}_n \mathrm{Ad}_{h^{-1}}(v) - \mathrm{Ad}_{h^{-1}}(v)) \in \mathrm{Ad}_h(\mathfrak{n}) \subset \mathfrak{n}. \tag{86}$$

The greatest normal subgroup of G which is contained in H is obtained by the following procedure.

Proposition II.23 *Suppose G is connected and $H \subset G$ a closed Lie subgroup. Define the decreasing sequence of subgroups of H:*

$$N_0 = H, \tag{87}$$

$$\forall\, i \geq 0,\ N_{i+1} = \left\{n \in H \mid \mathrm{Ad}_n\, v - v \in \mathfrak{n}_i,\ \forall\, v \in \mathfrak{g}\right\}. \tag{88}$$

Then, each $N_i \subset H$ is a closed normal subgroup of H and the intersection

$$N_\infty = \bigcap_i N_i \subset H \tag{89}$$

is the largest normal subgroup of G contained in H.

Proof The fact that N_i and N_∞ are normal will depend on the following computation, related to the preceding paragraph. Let $n \in G$, $g \in G$ and $k \geq 0$. Assume that $\mathrm{Ad}_n\, v = v + w(v)$ for any $v \in \mathfrak{g}$, with a corresponding $w(v) \in \mathfrak{n}_k$. Then

$$\mathrm{Ad}_{gng^{-1}}\, v = \mathrm{Ad}_g \mathrm{Ad}_n(\mathrm{Ad}_{g^{-1}}\, v) \tag{90}$$

$$= \mathrm{Ad}_g\left(\mathrm{Ad}_{g^{-1}}\, v + w(\mathrm{Ad}_{g^{-1}}(v))\right) \tag{91}$$

$$= v + \mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))). \tag{92}$$

Now, to see that each group N_i is normal in H, note that if $n \in N_i$ and $g \in H$ then the preceding computation shows that gng^{-1} belongs to N_i if, and only if, $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_{i-1}$. By hypothesis, $w(\mathrm{Ad}_{g^{-1}}(v)) \in \mathfrak{n}_{i-1}$. By recurrence, $\mathrm{Ad}_g(\mathfrak{n}_{i-1}) \subset \mathfrak{n}_{i-1}$, showing that we have indeed $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_{i-1}$.

It is clear that N_∞ is well defined and is normal in H. We have to show it is also normal in G. First, $\mathfrak{n}_\infty \subset \mathfrak{g}$ is an ideal. Indeed, by differentiation of $\mathrm{Ad}_n(v) = v + w(v)$ along a path $n(t)$ we have $[n, v] = w'(v)$ and it belongs to to $\mathfrak{n}_\infty$ since $w(v)$ does.

Since $\mathfrak{n}_\infty \subset \mathfrak{g}$ is an ideal and G is connected, the component of the identity of N_∞ is normal in G. But then it implies $\mathrm{Ad}_{g^{-1}}\, \mathfrak{n}_\infty = \mathfrak{n}_\infty$. By the preceding computation it implies $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_\infty$ and therefore that N_∞ is indeed normal.

To complete the proof, we show that for a normal subgroup $N \subset G$ contained in H, $N \subset N_\infty$: by induction, $N \subset H$ and if $N \subset N_i$ so $\mathfrak{n} \subset \mathfrak{n}_i$ and therefore $N \subset \left\{n \in H \mid \mathrm{Ad}_n\, v - v \in \mathfrak{n}_i,\ \forall\, v \in \mathfrak{g}\right\} = N_{i+1}$. ▒

III — Principal bundles

Consider a smooth right free action

$$\mu \colon P \times H \to P \tag{93}$$

of a Lie group H on a manifold P. We denote R_h the right action of H:

$$\forall\, h \in H, \forall\, p \in P,\ R_h(p) = \mu(p, h). \tag{94}$$

Such an action μ is called proper if for any K_1, K_2 compact subsets of P, the set

$$\{h \in H \mid R_h(K_1) \cap K_2 \neq \emptyset\} \tag{95}$$

is compact.

Let M be a manifold and H a Lie group. A (right) principal bundle

$$\pi\colon P \to M \tag{96}$$

consists of a manifold P with a right action μ by H which is locally trivial: for each $x \in M$, there exists a trivialization over an open set U containing x

$$\Psi = (\pi, \psi_H)\colon \pi^{-1}(U) \to U \times H \tag{97}$$

that is a diffeomorphism and such that

$$\Psi(\mu(u,h)) = (\pi(u), \psi_H(u)h). \tag{98}$$

A characterization of right actions which give rise to principal bundles is the following.

Proposition III.1 *Let $\mu\colon P \times H \to P$ be a proper smooth right free action. Then P/H is a smooth manifold with the quotient topology and it has a unique smooth structure such that the projection $P \to P/H$ defines a right H-principal bundle.*

Example Homogenous spaces are an important class of examples

$$\pi\colon G \times H \to G/H \tag{99}$$

where the right action $\mu\colon G \times H \to G$ is the Lie group law:

$$\mu(g,h) = gh. \tag{100}$$

This action is indeed proper. For if $h_i \in H$ and $K_1, K_2 \subset G$ are compact, assume that $R_{h_i}K_1 \cap K_2 \neq \emptyset$. We need to prove that h_i converge (up to a subsequence). For each i, we have necessarily $k_i^1 \in K_1$ and $k_i^2 \in K_2$ such that $R_{h_i}k_i^1 = k_i^2$. But both k_i^1 and k_i^2 converge (up to a subsequence) to k_1 and k_2 respectively. Hence $h_i = (k_i^1)^{-1}k_i^2$ converge (up to a subsequence) to $k_1^{-1}k_2$. The limit lies in H since it is closed.

Definition III.2 *Let $\pi_1\colon P_1 \to M_1$ and $\pi_2\colon P_2 \to M_2$ be two right H-principal bundles. A H-bundle diffeomorphism $F\colon P_1 \to P_2$ is a diffeomorphism that preserves the fibers and verifies $F \circ R_h = R_h \circ F$ (it is right equivariant).*

Since a H-bundle diffeomorphism preserves the fibers, it defines a diffeomorphism $f\colon M_1 \to M_2$. Hence, following diagram commutes.

$$\begin{array}{ccc} P_1 & \xrightarrow{F} & P_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array} \tag{101}$$

III.1 Ehresmann connections

Invariant vector fields With a right principal bundle P, one can consider a canonical vector field V^* associated to any $v \in \mathfrak{h}$:

$$V^*(p) = \frac{\mathrm{d}}{\mathrm{d}t} R_{\exp(tv)}p \Big|_{t=0} = R_{\exp(tv)_*}(0,1)\big|_{(p,0)}. \tag{102}$$

An alternative definition of V^* is the following. With $\mu\colon P \times H \to P$ the right action, we have

$$V^*(p) = \mu_*|_{(p,e)}(0,v). \tag{103}$$

For instance, in the case where $P = G$ is a Lie group and $H \subset G$, we get that V^* is again the left-invariant vector field $V^*(p) = L_{p_*}(v)$.

Definition III.3 *An Ehresmann connection ω_H on P is a $\mathfrak{h}$-valued 1-form satisfying:*

(1) for any $h \in H$, $R_h^*\omega_H = \mathrm{Ad}(h^{-1})\omega_H$;

(2) for any $v \in \mathfrak{h}$, $\omega_H(V^*) = v$.

This definition restricts to the Maurer-Cartan form in the case where M collapses to one point.

Note An equivalent formulation arises if we consider the distribution D defined by the kernel of ω_H. That distribution is an invariant *horizontal* distribution as $R_{h*}D = D$.

Lifting curves An Ehresmann connection defines a way to make a parallel displacement along curves of M from the fiber at the origin of the curve to the fiber at the end of the curve.

Let $\gamma\colon [0,1] \to M$ be a smooth path in M. Then there exists a unique lift $\tilde{\gamma}\colon [0,1] \to P$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}(t) \in \ker \omega_{\tilde{\gamma}(t)} \tag{104}$$

with an initial condition $\tilde{\gamma}(0) = p$.

Lemma III.4 *Both conditions $R_h^*\omega_H = \mathrm{Ad}(h^{-1})\omega_H$ and $\omega_H(V^*) = v$ are equivalent to the following.*

$$R_\psi^*\omega_H = \psi^*\theta_H + \mathrm{Ad}(\psi)^{-1}\omega_H, \tag{105}$$

where θ_H is the Maurer-Cartan form of H and ψ is any smooth function with values in H.

To be entirely precise, if $\psi\colon X \to H$ is a smooth map, then $R_\psi\colon P \times X \to P \times H \to P$ and we state

$$R_\psi^*\omega_H(u,v)|_{(p,x)} = \psi^*\theta_H(v)|_x + \mathrm{Ad}(\psi(x))^{-1}\omega_H(u)|_p. \tag{106}$$

Proof Since $R_\psi^*\omega$ is a differential form, we can consider separately vectors $(u,0)$ and $(0,v)$ at $(p,x) \in P \times X$. Since $R_\psi = \mu \circ (\mathrm{id} \times \psi)$ we only need to show the equivalence with:

$$\mu^*\omega(u,v)|_{(p,h)} = \theta_H(v) + \mathrm{Ad}(h)^{-1}\omega(u) \tag{107}$$

since the precomposition by $(\mathrm{id} \times \psi)_*$ would follow.

With vectors $(u,0)|_{(p,h)}$, the product $\mu_*(u,0)$ is equal to $R_{h*}(u)$. Hence the preceding formula and the first condition are equivalent.

With vectors $(0,v)|_{(p,h)}$, the product $\mu_*(0,v)$ gives exactly $V^*(\mu(p,h))$ where V^* is the invariant vector field corresponding to $\theta_H(v)$. Hence the preceding formula and the second condition are equivalent. ▧

III.2 Frame and coframe bundles

III.2.1 Some linear algebra

The linear group of matrices $\mathrm{GL}(n,\mathbf{R})$ does not act canonically on a vectorial space. Indeed, an isomorphism $\mathrm{GL}(V) \simeq \mathrm{GL}(n,\mathbf{R})$ relies on the choice of one (or two) basis of V.

However, $\mathrm{GL}(n,\mathbf{R})$ does act canonically on the spaces of the frames and coframes of V. Let

$$F = \{(e_1,\ldots,e_n) \text{ is an ordered basis of } V\}. \tag{108}$$

We say that F is the space of the frames of V.

We have a left action and a right action of $\mathrm{GL}(n,\mathbf{R})$ on F. It is defined by:

$$e'_i = g_i^j e_j = e_j g_i^j \tag{109}$$

where (g_i^j) is a matrix $g \in \mathrm{GL}(n,\mathbf{R})$. (We assume the Einstein summation convention.)

This left or right action on F corresponds to a right or left action on F^*, the space of the coframes:

$$F^* = \left\{(e^1,\dots,e^n) \text{ is an ordered basis of } V^*\right\}. \tag{110}$$

This last action is given by:

$$e^{i\prime} = e^j b_j^i = b_j^i e^j \tag{111}$$

with (b_j^i) a matrix $b \in \mathrm{GL}(n,\mathbf{R})$. The correspondance with the action on F is determined by the relation $e^{i\prime}(e_j) = \delta_j^i$:

$$e^{i\prime}(e_j) = e^k b_k^i (e_m g_j^m) = b_k^i g_j^k \tag{112}$$

and the equation $b_k^i g_j^k = \delta_j^i$ shows that $b = g^{-1}$ in $\mathrm{GL}(n,\mathbf{R})$.

The fact that the action on F by g becomes an action by g^{-1} on F^* shows that if g acts on the left (or respectively on the right) on F then g^{-1} acts on the right (or respectively on the left) on F^*. (Indeed, observe $(gh)^{-1} = h^{-1}g^{-1}$.)

III.2.2 Bundles and the tautological form

Definition III.5 *The frame bundle on a smooth manifold M is the set*

$$F = \left\{(x,\omega) \,\middle|\, x \in M \text{ and } \omega \text{ is a frame of } \mathrm{T}_x M\right\}. \tag{113}$$

And the coframe bundle is:

$$F^* = \left\{(x,\omega) \,\middle|\, x \in M \text{ and } \omega \text{ is a coframe of } \mathrm{T}_x M\right\}. \tag{114}$$

By the preceding considerations, each bundle F and F^* is a left *and* right principal $\mathrm{GL}(n,\mathbf{R})$-bundle.

Note A reduction of the principal bundle F and F^* to a subbundle (not necessarily principal) corresponds generally to the choice of a geometric structure on M.

Definition III.6 *A H-structure on a smooth manifold M is a principal subbundle with fiber a closed subgroup $H \subset \mathrm{GL}(n,\mathbf{R})$.*

Examples

(1) A Riemannian geometry on M, that is to say a Riemannian metric, corresponds to the choice of a subbundle of orthonormal frames or coframes.
(2) A conformal geometry on M, that is to say a conformal class of Riemannian metrics, corresponds to the choice of a subbundle of frames that are orthonormal up to an homogeneous factor.
(3) A contact structure on a 3-manifold M, that is to say the data of an everywhere non-integrable plane distribution $D^2 \subset \mathrm{T}M$ corresponds to the choice of a subbundle constituted of vectors (v_1, v_2, v_3) such that (v_1, v_2) generates D^2.

As a matter of fact, those three subbundles are principal. The first for the choice of $\mathrm{O}(n) \subset \mathrm{GL}(n,\mathbf{R})$, the second with $\mathbf{R}_+\mathrm{O}(n) \subset \mathrm{GL}(n,\mathbf{R})$ and the last with $P_2 \subset \mathrm{GL}(3,\mathbf{R})$ the set of the matrices:

$$P_2 = \left\{ \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{pmatrix} \right\} \subset \mathrm{GL}(3,\mathbf{R}). \tag{115}$$

With $\pi\colon P\to M$ a frame bundle, the previous identification $p=(x,\omega)$ allows to define a *tautological form* $\theta\colon \mathrm{T}P\to\mathbf{R}^n$ by:

$$\theta|_{(x,\omega)}(v)=\omega^{-1}\pi_*(v). \tag{116}$$

In contrast with the fact that the duality of $\mathbf{R}^n$ with its real forms is not canonical, the duality $\omega\mapsto\omega^{-1}$ of the frames with the coframes is canonical.

Consider the right action of H on P. Since $R_h\circ\pi=\pi$ we get $R_{h*}\pi_*=\pi_*$. Therefore:

$$R_h^*\theta|_{(x,\omega)}(v)=\theta|_{(x,\omega h)}(R_{h*}v)=h^{-1}\omega^{-1}\pi_*v \tag{117}$$

and in the Lie group $H\subset\mathrm{GL}(n,\mathbf{R})\subset\mathrm{Aff}(\mathbf{R}^n)$, the left action by h^{-1} is equivalent to the adjoint action by h^{-1}. Thus:

Lemma III.7 *Under the right action of H, the tautological form θ verifies:*

$$R_h^*\theta=\mathrm{Ad}(h)^{-1}\theta. \tag{118}$$

A section $\sigma\colon M\to P$ corresponds to the choice of a frame at each point $x\in M$. Any other frame α is then determined by a right translation:

$$\forall(x,\omega)\in P,\ \omega=\sigma(x)h_\sigma(x,\omega). \tag{119}$$

A section σ is also called a *moving frame.*

We can now describe Cartan's method to solve a first equivalence problem described as follows.

Let $P_1\to U_1$ and $P_2\to U_2$ be two frame bundles for a same subgroup $H\subset\mathrm{GL}(n,\mathbf{R})$. Assume that σ_1 and σ_2 are two sections. Under what condition a diffeomorphism $\phi\colon U_1\to U_2$ would be a *geometric diffeomorphism*? The fact that ϕ would preserve the geometry would mean that $\phi_*\sigma_1$ would still be a moving frame on U_2. This is equivalent by definition to

$$\phi_*\sigma_1=R_\psi\sigma_2 \tag{120}$$

$$\iff \mathrm{d}\phi_x\sigma_1(x)=\sigma_2(\phi(x))\psi(x) \tag{121}$$

for a certain smooth function $\psi\colon U_1\to H$.

Cartan's method is an interpretation of this equation at the level of the tautological forms θ_1 and θ_2.

Proposition III.8 *There exists a diffeomorphism $\phi\colon U_1\to U_2$ satisfying*

$$\phi_*\sigma_1=R_\psi\sigma_2 \tag{122}$$

for a function $\psi\colon U_1\to H$ if, and only if, there exists a H-bundle diffeomorphism $\tilde\phi\colon P_1\to P_2$ such that

$$\tilde\phi^*\theta_2=\theta_1. \tag{123}$$

Proof Assume that ϕ verifies $\phi_*\sigma_1=\sigma_2\psi$.

We define a lift $\tilde\phi$ by

$$\tilde\phi(x,\sigma_1(x)h)=\big(\phi(x),\mathrm{d}\phi_x\sigma_1(x)h\big) \tag{124}$$

it is right equivariant, preserves the fibers and of course lifts ϕ. By lifting ϕ and preserving the fibers we have $\pi_{2*}\tilde\phi_*=\phi_*\pi_{1*}$.

Now,

$$\begin{aligned}
\tilde\phi^*\theta_2\big|_{(x,\sigma_1h)} &= \theta_2|_{(\phi(x),\mathrm{d}\phi_x\sigma_1(x)h)}\,\mathrm{d}\tilde\phi\big|_{(x,\sigma_1h)} &(125)\\
&= h^{-1}\sigma_1(x)^{-1}\,\mathrm{d}\phi_x^{-1}\pi_{2*}\,\mathrm{d}\tilde\phi\big|_{(x,\sigma_1h)} &(126)\\
&= h^{-1}\sigma_1(x)^{-1}\,\mathrm{d}\phi_x^{-1}\,\mathrm{d}\phi_x\pi_{1*} &(127)\\
&= h^{-1}\sigma_1(x)^{-1}\pi_{1*} &(128)\\
&= \theta_1|_{(x,\sigma_1h)}, &(129)
\end{aligned}$$

hence $\tilde\phi^*\theta_2=\theta_1$.

Conversely, a H-bundle diffeomorphism $\widetilde{\phi}$ would induce a diffeomorphism $\phi\colon U_1 \to U_2$. But then $\widetilde{\phi}$ would send $(x, \sigma_1(x))$ on a point $(\phi(x), \sigma_2(\phi(x))\psi(x))$

$$\theta_1|_{(x,\sigma_1(x)}(v) = \widetilde{\phi}^* \theta_2\big|_{(x,\sigma_1(x))}(v) \tag{130}$$

$$\iff \sigma_1(x)^{-1}\pi_{1*}v = \psi(x)^{-1}\sigma_2(\phi(x))^{-1}\pi_{2*}(\mathrm{d}\widetilde{\phi})_{(x,\sigma_1(x))}(v) \tag{131}$$

$$= \psi(x)^{-1}\sigma_2(\phi(x))^{-1}\,\mathrm{d}\phi_x \pi_{1*}v \tag{132}$$

$$\iff \sigma_1(x)^{-1} = \psi(x)^{-1}\sigma_2(\phi(x))^{-1}\,\mathrm{d}\phi_x \tag{133}$$

$$\iff \mathrm{d}\phi_x \sigma_1(x) = \sigma_2(\phi(x))\psi(x). \tag{134}$$

And this is equivalent to $\phi_* \sigma_1 = R_\psi \sigma_2$. ▨

III.3 Parallel translation

If one puts together an Ehresmann connection on a frame bundle, then one has the very first example of a Cartan connection. Those would be defined later in generality. But we can make the following observation without directly mentioning Cartan connections.

Let $P \to M$ be a frame bundle. The tautological form θ expresses how a point in M moves relatively to the choice of a moving frame. But the way that the moving frame itself evolves is not measured by θ.

This corresponds to the fact that even if θ defines a basis of $\mathrm{T}_x M$ at every point, it is not the case in $\mathrm{T}_p P$.

If ω_H is an Ehresmann connection, then one gets a basis of $\mathrm{T}_p P$ by putting together both ω_H and θ. Indeed, note that θ vanishes along the fiber H (since it is a transformation combined with π_*) but ω_H is injective along $\mathrm{T}_p H$.

Let $\sigma = (X_1, \dots, X_n)$ be a moving frame around $x \in M$. It defines a section $p_\sigma(x) = (x, \sigma(x))$ of $P \to M$. Assume that ω_H is an Ehresmann connection.

Since $\sigma = (X_i)$ defines a section of $\pi\colon P \to M$ on an open $U \subset M$. One has an identification $\pi^{-1}(U) = U \times H$ by the right action of H on σ. In particular, any tangent vector $X \in \mathrm{T}U$ can be identified uniquely to $(X, 0) \in \mathrm{T}(U \times H)$.

Therefore, on a section, $\omega_H(X)$ is well defined, with the implicit mapping $X \mapsto (X, 0)$ allowing X to be a tangent vector in P. Note that this construction heavily depends on the choice of the section σ.

A parallel translation (in other words, an affined connection) is then determined by:

$$\nabla_X X_i = \omega_H(X) \cdot e_i \tag{135}$$

where $\omega_H(X)$ acts on X_i by the adjoint action.

By writing coordinates of the matrix group H, $\omega_H = \sum \omega^k_m e_k \otimes e^m$ and

$$\nabla_X X_i = \sum \omega^j_i(X) X_j. \tag{136}$$

IV — Cartan geometries

Definition IV.1 *A Cartan geometry is modeled on* $(\mathfrak{g}, \mathfrak{h})$ *is a right H-principal bundle $P \to M$ together with a 1-form $\omega_P\colon \mathrm{T}P \to \mathfrak{g}$, called a Cartan connection, verifying:*

(1) at each $p \in P$, ω_P is an isomorphism $\mathrm{T}_p P \to \mathfrak{g}$;

(2) for all $h \in H$, $R_h^* \omega_P = \mathrm{Ad}(h)^{-1}\omega_P$;

(3) for all $v \in \mathfrak{h}$ and V^* the corresponding invariant vector field, $\omega_P(V^*) = v$.

Note A Cartan connection furnishes a parallelism of P since $\mathrm{T}_p P \simeq \mathfrak{g}$ by ω_P. Hence $\mathrm{T}P \simeq P \times \mathfrak{g}$.

Definition IV.2 *The homogeneous space G/H is reductive if there exists a linear decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{137}$$

such that $\mathfrak{p}$ is $\mathrm{Ad}(H)$-invariant.

Note In such case, one can identify $\mathfrak{p}$ with the tangent space at M. Indeed, for $p \in P$, with $\mathrm{T}_p P \simeq \mathfrak{h} \oplus \mathfrak{p}$, the subspace $\mathfrak{p}$ is preserved by the right action of the fiber since $R_h^* \mathfrak{p} = \mathrm{Ad}(h)^{-1} \mathfrak{p} = \mathfrak{p}$. Hence, by pushing with π_*, it furnishes an identification of $\mathrm{T}_{\pi(p)} M$ with $\mathfrak{p}$.

However, even if $\mathfrak{p}$ can be identified with $\mathrm{T}_{\pi(p)} M$, this identification is not invariant by the right action on the fiber. It acts by $\mathrm{Ad}(H)^{-1}$ that is not trivial in general.

When a homogenous space is reductive, one can decompose a Cartan connection ω that has values in $\mathfrak{g}$ along $\mathfrak{h}$ and $\mathfrak{p}$, that is to say $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{p}}$. The factor $\omega_{\mathfrak{h}}$ is then an Ehresmann connection.

By the same proof as in the case of an Ehresmann connection (see III.4 (p. 15)), we have:

Lemma IV.3 *The two last conditions are equivalent to:*

$$R_\psi^* \omega_P = \psi^* \theta_H + \mathrm{Ad}(\psi)^{-1} \omega_P, \tag{138}$$

where θ_H is the Maurer-Cartan form of H and ψ is any smooth function with values in H.

Definition IV.4 *The curvature of a Cartan geometry is*

$$\Omega(u, v) = \mathrm{d}\omega(u, v) + [\omega(u), \omega(v)]. \tag{139}$$

If $\Omega = 0$ on $\mathrm{T}P$, then we say that the Cartan geometry is flat.

Homogeneous spaces The simplest example of a Cartan geometry is the fiber bundle $G \to G/H$ equipped with its Maurer-Cartan form θ_G. In this case $\Omega = \mathrm{d}\theta_G + \frac{1}{2}[\theta_G \wedge \theta_G] = 0$ is the structural equation.

Lemma IV.5 *If ψ is any smooth functions with values in H then*

$$R_\psi^* \Omega = \mathrm{Ad}(\psi)^{-1} \Omega. \tag{140}$$

Proof As noted previously, we know $R_\psi^* \omega_P$.

$$R_\psi^* \Omega = R_\psi^* \left(\mathrm{d}\omega_P + \frac{1}{2} [\omega_P \wedge \omega_P] \right) \tag{141}$$

$$= \mathrm{d} R_\psi^* \omega_P + \frac{1}{2} \left[R_\psi^* \omega_P \wedge R_\psi^* \omega_P \right] \tag{142}$$

$$= \psi^* \, \mathrm{d}\theta_H + \frac{1}{2} \left[\psi^* \theta_H \wedge \psi^* \theta_H \right] + \mathrm{Ad}(\psi)^{-1} \left(\mathrm{d}\omega_P + \frac{1}{2} [\omega \wedge \omega_P] \right) \tag{143}$$

$$= \psi^* 0 + \mathrm{Ad}(\psi)^{-1} \Omega \tag{144}$$

▨

Lemma IV.6 *The curvature $\Omega(u, v)$ vanishes if u or v is tangent to the fiber (belongs to $\mathrm{T}_p H \subset \mathrm{T}_p P$).*

Proof Assume that $u \in \mathrm{T}_p H$. Let $\psi\colon P \to H$ be such that $\psi(p) = e$ and $\psi_*(u) = -\omega_P u$. Then

$$R_\psi^* \omega_P(u)\Big|_p = \psi^* \theta_H + \mathrm{Ad}(\psi(p))^{-1} \omega_P(u) \tag{145}$$

$$= -\omega_P(u) + \omega_P(u) = 0 \tag{146}$$

hence $R_{\psi_*}(u) = 0$ and we get

$$\mathrm{Ad}(\psi)^{-1} \Omega(u,v) = \Omega(R_{\psi_*} u, R_{\psi_*} v) = \Omega(0, R_{\psi_*} v) = 0. \tag{147}$$

IV.1 Example 1: Riemannian geometry

As one can anticipate, a Cartan connection on a Riemannian geometry will coincide with the consideration of a certain Ehresmann connection together with the tautological form. But as usual in Riemannian geometry, one can consider many connections. Only one will have vanishing torsion. It is that property that will determine the corresponding Ehresmann connection.

We start with a description of $\mathrm{Eucl}(n)$, the group of the isometries of the Euclidean space. This space is the model for the Riemannian geometry. By the identification $\mathrm{Eucl}(n) = \mathbf{R}^n \rtimes \mathrm{O}(n)$, one can represent $\mathrm{Eucl}(n) \to \mathrm{GL}(n+1, \mathbf{R})$ by

$$(x, f) \mapsto \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix}. \tag{148}$$

So the Lie algebra is:

$$\mathfrak{eucl}(n) = \begin{pmatrix} \mathfrak{o}(n) & \mathbf{R}^n \\ 0 & 1 \end{pmatrix}. \tag{149}$$

An important observation is that the adjoint action by an orthonormal element is a left translation on the $\mathbf{R}^n$ coordinate:

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AbA^T & Av \\ 0 & 1 \end{pmatrix}. \tag{150}$$

The adjoint action on the Lie algebra is:

$$\left[\begin{pmatrix} a & u \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & v \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [a,b] & av - bu \\ 0 & 0 \end{pmatrix}. \tag{151}$$

The Maurer-Cartan form θ_{Eucl} can be written:

$$\theta_{\mathrm{Eucl}} = g^{-1} \mathrm{d}g = \begin{pmatrix} \theta_{\mathrm{O}(n)} & \theta_{\mathbf{R}^n} \\ 0 & 0 \end{pmatrix}. \tag{152}$$

And the structural equation becomes

$$\begin{pmatrix} \mathrm{d}\theta_{\mathrm{O}(n)} + \frac{1}{2}\left[\theta_{\mathrm{O}(n)} \wedge \theta_{\mathrm{O}(n)}\right] & \mathrm{d}\theta_{\mathbf{R}^n} + \left[\theta_{\mathrm{O}(n)} \wedge \theta_{\mathbf{R}^n}\right] \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{153}$$

We will describe the (Levi-Civita) Cartan connection ω_P decomposed as:

$$\omega_P = \begin{pmatrix} \omega_{\mathrm{O}(n)} & \omega_{\mathbf{R}^n} \\ 0 & 0 \end{pmatrix} \tag{154}$$

And since $\mathrm{Eucl}(n)/\mathrm{O}(n)$ is a frame bundle, we start by setting $\omega_{\mathbf{R}^n} = \theta$ the tautological form.

Along any local section $\sigma\colon U \to P$, we get a trivialisation $\pi^{-1}(U) = U \times H$. Since θ vanishes on vertical vectors, that is to say on vectors tangent to each fiber H, so does $\mathrm{d}\theta$. But θ is a base of forms on U. Therefore, there exist numbers a^i_{jk} such that

$$\mathrm{d}\theta^i = \sum_{jk} a^i_{jk} \theta^j \wedge \theta^k. \tag{155}$$

By exterior derivative, the numbers a^i_{jk} verify $a^i_{jk} = -a^i_{kj}$.

This construction is independent from the choice of the section. Since $\mathrm{d}\theta^i$ is a differential 2-form on $\mathrm{T}P$, the existence and unicity of the numbers a^i_{jk} is unambiguous.

The goal is to define a 1-form $\omega = \omega_{\mathrm{O}(n)}$ verifying

$$\mathrm{d}\theta = -[\omega \wedge \theta] \tag{156}$$

in order to verify the $\mathbf{R}^n$-factor of the structural equation. It is that condition that will prove the unicity of the (Levi-Civita) Cartan connection.

Let $\omega = \sum \omega^i_j e_i \otimes e^j$ be defined by the numbers

$$\omega^i_j = \sum_k \left(a^i_{jk} + a^j_{ki} - a^k_{ij} \right) \theta^k. \tag{157}$$

Then ω is indeed $\mathfrak{o}(n)$-valued since ω^i_j is anti-symmetric:

$$\omega^j_i = \sum_k \left(a^j_{ik} + a^i_{kj} - a^k_{ji} \right) \theta^k = \sum_k - \left(a^i_{jk} + a^j_{ki} - a^k_{ij} \right) \theta^k. \tag{158}$$

And it verifies $\mathrm{d}\theta = -[\omega \wedge \theta]$ since:

$$\sum_j -\omega^i_j \wedge \theta^j = \sum_j \theta^j \wedge \omega^i_j = \sum_{jk} \left(a^i_{jk} + a^j_{ki} - a^k_{ij} \right) \theta^j \wedge \theta^k \tag{159}$$

$$= \sum_{jk} a^i_{jk} \theta^j \wedge \theta^k + \sum_{j<k} \left(\left(a^j_{ki} - a^k_{ij} \right) - \left(a^k_{ji} - a^j_{ik} \right) \right) \theta^j \wedge \theta^k \tag{160}$$

$$= \mathrm{d}\theta^i + 0 = \mathrm{d}\theta^i. \tag{161}$$

The solution ω is unique. Indeed, if γ were another solution verifying $\mathrm{d}\theta^i = -\gamma^i_j \wedge \theta^j$, then $\omega - \gamma$ would again be solution and it would show that $(\omega^i_j - \gamma^i_j) \wedge \theta^j = 0$. But a transformation $\alpha \in \mathfrak{o}(n)$ acts trivially by the adjoint action on $\mathbf{R}^n$ if and only if $\alpha = 0$. In this case, it gives $\omega^i_j - \gamma^i_j = 0$.

Now that we have ω, we can show that together with θ it defines a Cartan connection. In fact, this connection is exactly the Levi-Civita connection.

Lemma IV.7 *The form*

$$\omega_P = \begin{pmatrix} \omega & \theta \\ 0 & 0 \end{pmatrix} \tag{162}$$

is a Cartan connection.

Proof The form θ is certainly surjective on $\mathbf{R}^n \subset \mathfrak{eucl}(n) = \mathfrak{o}(n) \oplus \mathbf{R}^n$. If we show that $\omega_P(V^*) = v$ for any $v \in \mathfrak{o}(n)$ then we will have shown that ω_P is a linear isomorphism at each T_pP. So we need to prove the two last conditions to be a Cartan connection.

First let $h \in \mathrm{O}(n)$. We have proven by lemma III.7 (p. 17) that $R_h^* \theta = \mathrm{Ad}(h)^{-1}\theta$. So we need to prove this statement on ω.

$$0 = R_h^* (\mathrm{d}\theta + [\omega \wedge \theta]) = R_h^* \mathrm{d}\theta + \left[R_h^* \omega \wedge R_h^* \theta \right] \tag{163}$$

$$= \mathrm{dAd}(h)^{-1}\theta + \left[R_h^* \omega \wedge \mathrm{Ad}(h)^{-1}\theta \right] \tag{164}$$

$$= \mathrm{Ad}(h)^{-1}\, \mathrm{d}\theta + \mathrm{Ad}(h)^{-1} \left[\mathrm{Ad}(h) R_h^* \omega \wedge \theta \right] \tag{165}$$

$$= \mathrm{d}\theta + \left[\mathrm{Ad}(h) R_h^* \omega \wedge \theta \right] \tag{166}$$

But ω is the unique solution to $\mathrm{d}\theta + [\omega \wedge \theta] = 0$. Hence

$$\mathrm{Ad}(h) R_h^* \omega = \omega \iff R_h^* \omega = \mathrm{Ad}(h)^{-1} \omega. \tag{167}$$

Next, we prove that $\omega_P(V^*) = v$ for any $v \in \mathfrak{o}(n)$. This statement is equivalent to $\omega(V^*) = v$ since θ takes its values in $\mathbf{R}^n$. Recall that:

$$V^*(p) = \frac{\mathrm{d}}{\mathrm{d}t} R_{\exp(tv)} p \bigg|_{t=0} = R_{\exp(tv)*}(0,1)\big|_{(p,0)}. \tag{168}$$

Let $\psi(t) = \exp(tv)$. We redo the previous computation. This time:

$$\mathrm{dAd}(\psi)^{-1}\theta = \mathrm{d}(\mathrm{Ad}(\psi)^{-1}) \wedge \theta + \mathrm{Ad}(\psi)^{-1} \wedge \mathrm{d}\theta \tag{169}$$

$$= \mathrm{Ad}(\psi)^{-1}\left[-\psi^*\theta_{\mathrm{O}(n)} \wedge \theta\right] + \mathrm{Ad}(\psi)^{-1} \wedge \mathrm{d}\theta \tag{170}$$

hence

$$0 = \mathrm{dAd}(\psi)^{-1}\theta + \left[R_\psi^*\omega \wedge \mathrm{Ad}(\psi)^{-1}\theta\right] \tag{171}$$

$$= \mathrm{Ad}(\psi)^{-1}\left(\mathrm{d}\theta - \left[\mathrm{Ad}(\psi)R_\psi^*\omega - \psi^*\theta_{\mathrm{O}(n)} \wedge \theta\right]\right). \tag{172}$$

It gives:

$$R_\psi^*\omega = \psi^*\theta_{\mathrm{O}(n)} + \mathrm{Ad}(\psi)^{-1}\omega \tag{173}$$

hence when $t = 0$:

$$R_\psi^*\omega\Big|_{t=0}(0,1) = \omega(V^*) \tag{174}$$

$$= \theta_{\mathrm{O}(n)}(\psi'(0)) + \mathrm{Ad}(\psi(0))^{-1}\omega(0) = v. \tag{175}$$

▧

Now that we have the (Levi-Civita) connection, we can compute its curvature. Recall that by construction, the $\mathbf{R}^n$ part of the curvature vanishes since $\mathrm{d}\theta + [\omega\wedge\theta] = 0$. The curvature is therefore:

$$\Omega = \begin{pmatrix} \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] & 0 \\ 0 & 0 \end{pmatrix}. \tag{176}$$

Recall that Ω vanishes on the vectors tangent to the fiber. So it only depends on the tautological form θ. Hence there exists numbers R^i_{jkl} such that

$$\Omega = \sum_{ij} W^i_j e_i \otimes e^j, \tag{177}$$

$$W^i_j = \sum_{kl} R^i_{jkl}\theta^k \wedge \theta^l. \tag{178}$$

IV.2 Example 2: web geometry

Web geometries on $\mathbf{R}^2$ are a way to study the geometry of differential equations

$$\mathrm{d}y = F(x,y)\,\mathrm{d}x. \tag{179}$$

Definition IV.8 *A web on $\mathbf{R}^2$ is the data of three line distributions $L_1, L_2, L_3 \subset \mathrm{T}\mathbf{R}^2$ such that any two are linearly independent at each point.*

By duality, a line corresponds to a kernel of a form: $L_1 = \ker\alpha^1$. Since lines are not parametrized by specific vectors, α^1 and $\lambda\alpha^1$ generate the same line L_1.

Hence, by rescaling, since $\alpha^3 = \lambda\alpha^1 + \mu\alpha^2$, we can assume that $\alpha^3 = \alpha^1 - \alpha^2$.

Definition IV.9 *A coframe of a web on $\mathbf{R}^2$ is the data of three 1-forms $\alpha^1, \alpha^2, \alpha^3 \in \mathrm{T}^*\mathbf{R}^2$ such that any two are linearly independent at each point and $\alpha^3 = \alpha^1 - \alpha^2$.*

Hence a coframe is in fact the data of only α^1 and α^2. So it is indeed a coframe of $\mathbf{R}^2$. The bundle of all coframes of a web on $\mathbf{R}^2$ is a $\mathbf{R}^*$-principal bundle.

Indeed, if α^1 becomes $\lambda_1\alpha^1$ and α^2 becomes $\lambda_2\alpha^2$ then $\lambda_1\alpha^1 - \lambda_2\alpha^2$ must still be proportional to α^3, hence $\lambda_1 = \lambda_2$.

Flat model A flat model for this geometry is given by $\alpha^1 = \mathrm{d}y$, $\alpha^2 = \mathrm{d}x$ and the third form $\alpha^3 = \mathrm{d}y - \mathrm{d}x$. We admit that the invariance group is exactly $G = \mathbf{R}^2 \rtimes \mathbf{R}^*$

where $\mathbf{R}^2$ acts by translation and the isotropy $H = \mathbf{R}^*$ by dilation. A representation of $G \to \mathrm{GL}(3, \mathbf{R})$ is given by:

$$(x, y, \lambda) \mapsto \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ x & y & 1 \end{pmatrix}. \tag{180}$$

One can check that in this representation,

$$\mathrm{Ad}(\lambda^{-1})(x, y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x\lambda & y\lambda & 0 \end{pmatrix}, \tag{181}$$

$$[\lambda \wedge (x, y)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x \wedge \lambda & y \wedge \lambda & 0 \end{pmatrix}. \tag{182}$$

The torsion free Cartan connection With any section α we define the tautological form θ on $\mathrm{T}P$ with $P \to \mathbf{R}^2$ the $\mathbf{R}^*$-coframe bundle by

$$\theta|_{\alpha\lambda}(v) = \alpha(\pi_*(v))\lambda = (\pi^* \alpha\lambda)(v). \tag{183}$$

Since $\alpha = (\alpha^1, \alpha^2)$ is defined on $\mathrm{T}\mathbf{R}^2$,

$$\mathrm{d}\alpha = \alpha \wedge (\tau^1, \tau^2) = (\alpha^1 \wedge \tau^1, \alpha^2 \wedge \tau^2) \tag{184}$$

where τ^1 and τ^2 are two 1-forms defined on $\mathrm{T}\mathbf{R}^2$. Hence each $\tau^i = \tau^i_1 \alpha^1 + \tau^i_2 \alpha^2$. Since we impose the value $\alpha^i \wedge \tau^i$, only one of two components of τ^i is determined and we can assume that $\tau^1 = \tau^2$ such that $\mathrm{d}\alpha = \alpha \wedge \tau$ with $\tau = (\tau^1, \tau^1)$.

Now, with $\theta = \pi^* \alpha\lambda$,

$$\begin{aligned} \mathrm{d}\theta &= \pi^* \left((\mathrm{d}\alpha)\lambda - \alpha \wedge \mathrm{d}\lambda\right) & (185) \\ &= \pi^* \left(\alpha\lambda \wedge \tau - \alpha\lambda \wedge \lambda^{-1} \mathrm{d}\lambda\right) & (186) \\ &= \theta \wedge \pi^* \tau - \theta \wedge \lambda^{-1} \mathrm{d}\lambda & (187) \\ &= \theta \wedge \left(\pi^* \tau - \lambda^{-1} \mathrm{d}\lambda\right) & (188) \end{aligned}$$

and we set

$$\omega = \lambda^{-1} \mathrm{d}\lambda - \pi^* \tau \tag{189}$$

and one should note that $\lambda^{-1} \mathrm{d}\lambda$ is the Maurer-Cartan form of the fiber $H = \mathbf{R}^*$.

Therefore, with

$$\omega_P = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ \theta^1 & \theta^2 & 0 \end{pmatrix} \tag{190}$$

we can check that ω_P is indeed a Cartan connection and is unique by verifying $\mathrm{d}\theta + [\omega \wedge \theta] = 0$.

The curvature form is determined by:

$$\Omega = \pi^* \begin{pmatrix} -\mathrm{d}\tau & 0 & 0 \\ 0 & -\mathrm{d}\tau & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{191}$$

Note that this curvature form is defined by the value of $-\mathrm{d}\tau$ on the basis and $-\mathrm{d}\tau = K\alpha^1 \wedge \alpha^2$. It is known as the Blaschke-Chern curvature.

Application Consider the equation

$$\mathrm{d}y = F(x, y)\, \mathrm{d}x \tag{192}$$

it furnishes a web

$$\alpha^1 = \mathrm{d}y, \tag{193}$$

$$\alpha^2 = F(x,y)\,\mathrm{d}x, \tag{194}$$

$$\alpha^3 = \mathrm{d}y - F(x,y)\,\mathrm{d}x = \alpha^1 - \alpha^2 \tag{195}$$

where $F(x,y)$ does not vanish.

By following the method, we differentiate (α^1, α^2). It gives:

$$\mathrm{d}\alpha^1 = 0 \tag{196}$$

$$\mathrm{d}\alpha^2 = \frac{\partial F}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x \tag{197}$$

$$= \frac{1}{F}\frac{\partial F}{\partial y}\,\mathrm{d}y \wedge \alpha^2 \tag{198}$$

and it determines τ by verifying $\mathrm{d}\alpha = \alpha \wedge \tau$:

$$\tau = -\frac{1}{F}\frac{\partial F}{\partial y}\,\mathrm{d}y. \tag{199}$$

Hence, the Blaschke-Chern curvature is:

$$-\mathrm{d}\tau = \frac{\partial}{\partial x}\left(\frac{1}{F}\frac{\partial F}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \tag{200}$$

$$= \frac{1}{F}\left(\frac{\partial^2 F}{\partial x \partial y} - \frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \tag{201}$$

$$= \frac{-1}{F^2}\left(\frac{\partial^2 F}{\partial x \partial y} - \frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\right)\alpha^1 \wedge \alpha^2. \tag{202}$$

For instance, it vanishes for any $F(x,y)$ linear in x and y, or any F independent from x or y. It does not vanish with $F(x,y) = \sin(xy)$. Indeed:

$$\frac{\partial^2 F}{\partial x \partial y} = -xy\sin(xy) + \cos(xy), \tag{203}$$

$$\frac{1}{F}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} = xy\frac{\cos(xy)^2}{\sin(xy)}. \tag{204}$$

IV.3 Example 3: path geometry

The model space for path geometry is ${}^{G}\!/_{H}$ where $G = SL(3,\mathbf{R})$ and $H = B$, the so called Borel subgroup of G of upper triangular matrices. It can be realized as a flag manifold F_{12}, the space of complete flags in $\mathbf{R}^3$.

Definition IV.10 *Le M be a real three dimensional manifold and* $\mathrm{T}M$ *be its tangent bundle.*

(1) A path structure *$\mathscr{L} = (E^1, E^2)$ on M is a choice of two line sub-bundles E^1 and E^2 in* $\mathrm{T}M$*, such that $E^1 \cap E^2 = \{0\}$ and $E^1 \oplus E^2$ is a contact distribution.*

(2) A strict path structure *$\mathscr{T} = (E^1, E^2, \theta)$ on M is a path structure with a fixed contact form θ such that* $\ker\theta = E^1 \oplus E^2$.

(3) A (local) automorphism *of $(M, \mathscr{T})$ is a (local) diffeomorphism f of M that preserves E^1, E^2 and θ.*

The condition that $E^1 \oplus E^2$ be a contact distribution means that, locally, there exists a one form θ on M such that $\ker\theta = E^1 \oplus E^2$ and $\theta \wedge d\theta$ is never zero. On the other hand, for strict path structures we impose the existence of a *globally defined* contact form θ. Therefore, strict path structures are *unimodular geometries*: there exists a canonical volume form $\mu_{\mathscr{T}} = \theta \wedge d\theta$ on M, preserved by the automorphism group of $\mathscr{T}$ (in contrast, path structures are not unimodular).

There exists a unique vector field R such that $d\theta(R,\cdot) = 0$ and $\theta(R) = 1$, called the *Reeb vector field* of θ, that we will also call the Reeb vector field of the strict path structure $\mathscr{T}$. In particular, the distribution $E^1 \oplus E^2$ of a strict path structure $\mathscr{T}$ is thus oriented, and the manifold M supporting $\mathscr{T}$ is orientable.[1]

Flat path model Flat path geometry is the geometry of real flags in $\mathbf{R}^3$. That is the geometry of the space of all couples (p,l) where $p \in \mathbf{R}P^2$ and l is a real projective line containing p. The space of flags is identified to the quotient

$$\mathrm{SL}(3,\mathbf{R})\big/B \tag{205}$$

where B is the Borel group of all real upper triangular matrices.

Flat strict path model The Heisenberg group Heis(3) is the flat model for the strict path geometry. With

$$\mathrm{Heis}(3) = \left\{(x,y,t) \in \mathbf{R}^3\right\} \tag{206}$$

and the multiplication defined by $(x_1,y_1,t_1)\cdot(x_2,y_2,t_2) = (x_1+x_2, y_1+y_2, t_1+t_2+2(x_1y_2-x_2y_1))$. We consider the left invariant distributions determined by their value at the origin:

$$E_1 = \frac{\partial}{\partial x} \text{ and } E_2 = \frac{\partial}{\partial y} \tag{207}$$

and it has a global corresponding contact form:

$$\theta = \mathrm{d}t - x\,\mathrm{d}y - y\,\mathrm{d}x. \tag{208}$$

IV.3.1 Path structures and second order differential equations

A second order differential equation in one variable is described locally as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right). \tag{209}$$

With $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, this defines a path structure on a neighborhood of a point in $\mathbf{R}^3$ with coordinates (x,y,p):

$$E_1 = \ker(\mathrm{d}y - p\,\mathrm{d}x) \cap \ker(\mathrm{d}p - F\,\mathrm{d}x), \tag{210}$$

$$E_2 = \ker(\mathrm{d}x) \cap \ker(\mathrm{d}y). \tag{211}$$

The contact structure is defined by the form

$$\theta = \mathrm{d}y - p\,\mathrm{d}x. \tag{212}$$

By defining the forms $Z^1 = \mathrm{d}x$ and $Z^2 = \mathrm{d}p - F\,\mathrm{d}x$, one has that $\mathrm{d}\theta = Z^1 \wedge Z^2$.

One can show that every path structure is, in fact, locally equivalent to a second order equation. That is, there exists local coordinates such that E_1 and E_2 are defined via a second order differential equation as above.

For, one first finds coordinates such that $E_2 = \ker \mathrm{d}x \cap \ker \mathrm{d}y$ by the flowbox theorem. Forms which annihilate $E_2 + E_1$ should be described by $q\,\mathrm{d}x + p\,\mathrm{d}y$, for functions q and p. Without loss of generality, one can assume locally that $\mathrm{d}x + p\,\mathrm{d}y$ and using the contact condition one concludes that x, y, p are local coordinates. Then $E_1 = \ker(\gamma\,\mathrm{d}p + \beta(\mathrm{d}y - G\,\mathrm{d}x)) \cap \ker(\mathrm{d}y - p\,\mathrm{d}x)$ and one lets, without loss of generality, $\beta = 0$ and $\alpha = 1$.

[1] If the contact distribution is oriented, then there exists a global contact form. Indeed, using a global metric on the distribution one can define locally a transversal vector to the distribution taking a Lie bracket of orthonormal vectors in the distribution. This defines a global 1-form.

Local equivalence (also called point equivalence) between path structures happens when there exists a local diffeomorphism which gives a correspondence between the lines defining each structure.

One can choose a contact form θ up to a scalar function and interpret this as follows: one has a $\mathbf{R}^*$-bundle over the manifold given by the choice of θ at each point (one might keep only positive multiples for simplicity). Over this line bundle one defines the tautological form $\omega_{\theta\alpha} = \pi^*\theta\alpha$. This bundle is trivial if and only if there exists a global contact form θ.

Let θ and local forms Z^1 and Z^2 defining the lines as above such that $\theta = Z^1 \wedge Z^2$. There exists global forms Z^1 and Z^2 if and only if there exists global vector fields along the lines. Clearly, if the contact distribution is oriented, it suffices that there exists a global vector field along one of the foliations by lines. Most of the three-dimensional Lie groups have left invariant path structures with global forms.

IV.3.2 Examples

Example 1 Consider the Heisenberg group

$$\mathrm{Heis}(3) = \{(z,t) \in \mathbf{C}\times\mathbf{R}\} \tag{213}$$

with multiplication defined by $(z_1,t_1)\cdot(z_2,t_2) = (z_1+z_2, t_1+t_2+2\,\mathrm{Im}\, z_1\overline{z_2})$. The contact form

$$\theta = \mathrm{d}t - x\,\mathrm{d}y - y\,\mathrm{d}x \tag{214}$$

is invariant under left multiplications (also called Heisenberg translations). If $\Lambda \subset \mathrm{Heis}(3)$ is a lattice then the quotient $\Lambda\backslash\mathrm{Heis}(3)$ is a circle bundle over the torus with a globally defined contact form.

A lattice Λ determines a lattice $\Gamma \subset \mathbf{C}$ corresponding to the projection in the exact sequence

$$\{0\} \to \mathbf{R} \to \mathrm{Heis}(3) \to \mathbf{C} \to \{0\}. \tag{215}$$

There are many global vector fields in the distribution defined by θ and invariant under Λ, it suffices to lift a vector field on $\mathbf{C}$ invariant under Γ. All circle bundles obtained in this way are not trivial and the fibers are transverse to the distribution.

Example 2 We consider the torus T^3 with coordinates $(x,y,t) \in {\mathbf{R}}/{\mathbf{Z}}^3$ and the global contact form

$$\theta_n = \cos(2\pi n t)\,\mathrm{d}x - \sin(2\pi n t)\,\mathrm{d}y. \tag{216}$$

There are two canonical global vector fields on the distribution given by

$$\frac{\partial}{\partial t} \text{ and } \sin(2\pi n t)\frac{\partial}{\partial x} + \cos(2\pi n t)\frac{\partial}{\partial y}. \tag{217}$$

In this example, the fiber given by the coordinate t has tangent space contained in the distribution.

Example 3 An homogeneous example is the Lie group SU(2) with left invariant vector fields X and Y with $Z = [X,Y]$ and cyclic commutation relations. The vector fields X and Y define a path structure on SU(2).

Example 4 Another homogeneous example is the Lie group SL(2, $\mathbf{R}$) with left invariant vector fields X and Y with $Z = [X,Y]$ with $[Z,X] = X$ and $[Z,Y] = -Y$ given by the generators in $\mathfrak{sl}(2)$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{218}$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{219}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{220}$$

The path structure defined by X and Y induces a path structure on the quotient $\Gamma \backslash \mathrm{SL}(2,\mathbf{R})$ by a discrete torsion free subgroup $\Gamma \subset \mathrm{SL}(2,\mathbf{R})$. This structure is invariant under the flow defined by right multiplication by e^{tZ}.

Example 5 Let Σ be a surface equipped with a Riemannian metric. The geodesic flow on its unit tangent bundle $\mathrm{T}^1\Sigma$ defines a distribution which, together with the distribution defined by the vertical fibers of the projection of the unit tangent bundle on Σ, defines a path structure which is not invariant under the geodesic flow. For $\Sigma = \mathbf{H}^2_{\mathbf{R}}$, the hyperbolic upper plane, we obtain $\mathrm{T}^1\Sigma = \mathrm{PSL}(2,\mathbf{R})$ with distributions defined by the left invariant distributions $X - Y$ and Z (using the same generators of the Lie algebra $\mathfrak{sl}(2)$ as in the previous example).

IV.3.3 Path structures with a fixed contact form

We now go back to strict path structures, by considering the specific case of Cartan geometries modeled on Heis(3), the flat model of strict path structures. So G denotes from now on the subgroup of $\mathrm{SL}(3,\mathbf{R})$ defined by

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ x & \frac{1}{a^2} & 0 \\ z & y & a \end{pmatrix} \middle| \, a \in \mathbf{R}^*, (x,y,z) \in \mathbf{R}^3 \right\} \tag{221}$$

and $H \subset G$ the isotropy subgroup of G defined by

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix} \right\}. \tag{222}$$

The Heisenberg group is identified to:

$$\mathrm{Heis}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \right\}. \tag{223}$$

The semidirect structure $G = \mathrm{Heis}(3) \rtimes H$ is described by the action of H on Heis(3) by conjugation:

$$\begin{pmatrix} \frac{1}{a} & & \\ & a^2 & \\ & & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{pmatrix} \begin{pmatrix} a & & \\ & \frac{1}{a^2} & \\ & & a \end{pmatrix} = \begin{pmatrix} 1 & & \\ a^3 x & & \\ z & \frac{1}{a^3} y & 1 \end{pmatrix}. \tag{224}$$

Writing the Maurer-Cartan form of G as the matrix

$$\begin{pmatrix} w & 0 & 0 \\ \theta^1 & -2w & 0 \\ \theta & \theta^2 & w \end{pmatrix} \tag{225}$$

one obtains the structural equations:

$$\begin{cases} \mathrm{d}\theta + \theta^2 \wedge \theta^1 = 0 \\ \mathrm{d}\theta^1 - 3w \wedge \theta^1 = 0 \\ \mathrm{d}\theta^2 + 3w \wedge \theta^2 = 0 \\ \mathrm{d}w = 0. \end{cases} \tag{226}$$

Let M be a three-manifold equipped with a strict path structure $\mathscr{T} = (E^1, E^2, \theta)$ with Reeb vector field R. Now let $X_1 \in E^1$, $X_2 \in E^2$ be such that $\mathrm{d}\theta(X_1, X_2) = 1$. The dual coframe of (X^1, X^2, R) is $(\alpha^1, \alpha^2, \theta)$, with two 1-forms α_1 and α_2 verifying $\mathrm{d}\theta = \alpha^1 \wedge \alpha^2$.

At any point $x \in M$, any coframe $(\theta^1, \theta^2, \theta)$ verifying $\mathrm{d}\theta = \theta^1 \wedge \theta^2$ is of the form

$$\theta^1 = a^3 \alpha^1,\ \theta^2 = \frac{1}{a^3}\theta^2 \tag{227}$$

with a a function with values in $\mathbf{R}^*$.

Definition IV.11 *We denote by $\pi \colon P \to M$ the right $\mathbf{R}^*$-coframe bundle over M given by the set of coframes $(\theta^1, \theta^2, \theta)$.*

We will denote the tautological forms defined by θ^1, θ^2 and θ by using the same letters. That is, we write $\theta^i = \pi^*\theta^i$.

Proposition IV.12 *There exists a unique Cartan connection on $P \to M$*

$$\omega = \begin{pmatrix} w & 0 & 0 \\ \theta^1 & -2w & 0 \\ \theta & \theta^2 & w \end{pmatrix} \tag{228}$$

such that its curvature form is of the form

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] = \begin{pmatrix} \mathrm{d}w & 0 & 0 \\ \theta \wedge \tau^1 & -2\mathrm{d}w & 0 \\ 0 & \theta \wedge \tau^2 & \mathrm{d}w \end{pmatrix} \tag{229}$$

with $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$.

Observe that the condition $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$ implies that we may write $\tau^1 = \tau^1_2 \theta^2$ and $\tau^2 = \tau^2_1 \theta^1$.

Proof We differentiate the tautological forms. One obtains with $\theta^1 = a^3 \alpha^1$:

$$\mathrm{d}\theta^1 = 3a^2\,\mathrm{d}a \wedge \alpha^1 + a^3\,\mathrm{d}\alpha^1 \tag{230}$$

$$= -3\theta^1 \wedge \frac{\mathrm{d}a}{a} + a^3\left(v^1 \wedge \alpha^1 + b_1 \theta \wedge \alpha^2\right) \tag{231}$$

for a certain function b_1 and a 1-form v^1 defined on M. Rearranging terms we obtain

$$\mathrm{d}\theta^1 = -3\theta^1 \wedge \left(\frac{\mathrm{d}a}{a} - v^1\right) + a^6 b_1 \theta \wedge \theta^2. \tag{232}$$

Analogously we have

$$\mathrm{d}\theta^2 = 3\theta^2 \wedge \left(\frac{\mathrm{d}a}{a} + v^2\right) + \frac{b_2}{a^6}\theta \wedge \theta^1. \tag{233}$$

Observe now that by differentiating $\mathrm{d}\theta = \alpha^1 \wedge \alpha^2$ one obtains that

$$\mathrm{d}^2\theta = 0 = \mathrm{d}\alpha^1 \wedge \alpha^2 - \alpha^1 \wedge \mathrm{d}\alpha^2 \tag{234}$$

$$= \alpha^1 \wedge v^1 \wedge \alpha^2 - \alpha^1 \wedge \alpha^2 \wedge v^2. \tag{235}$$

This implies that the term in θ of v^1 and v^2 only differ by a sign. One can therefore define a unique w by adding to $\frac{\mathrm{d}a}{a} - v^1$ the term in v^2 which is proportional to θ^2.

Unicity of this construction follows easily from Cartan's lemma. The verification that it is actually a Cartan connection is left to the reader. ▨

An example with constant curvature Consider $\mathrm{SL}(2,\mathbf{R})$ with its left invariant vector fields defined by a Lie algebra basis (E,F,H) of $\mathfrak{sl}(2)$ with $[E,F]=H$, $[H,E]=2E$ and $[H,F]=-2F$. Explicitly:

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix} \tag{236}$$

$$F=\begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{237}$$

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{238}$$

The structural equations of $\mathrm{SL}(2,\mathbf{R})$ for a dual basis α^1,α^2,θ are:

$$\begin{cases}\mathrm{d}\theta+\alpha^1\wedge\alpha^2=0\\ \mathrm{d}\alpha^1-2\alpha^1\wedge\theta=0\\ \mathrm{d}\alpha^2-2\theta\wedge\alpha^2=0.\end{cases} \tag{239}$$

Indeed, note that:

$$\begin{pmatrix}\theta&\alpha^1\\\alpha^2&-\theta\end{pmatrix}\wedge\begin{pmatrix}\theta&\alpha^1\\\alpha^2&-\theta\end{pmatrix}=\begin{pmatrix}\alpha^1\wedge\alpha^2&-2\alpha^2\wedge\theta\\-2\theta\wedge\alpha^2&-\alpha^1\wedge\alpha^2\end{pmatrix}. \tag{240}$$

Now, we define a strict path structure on $\mathrm{SL}(2,\mathbf{R})$. At any point, we do a left translation (by $\mathrm{SL}(2,\mathbf{R})$) of $(\mathbf{R}F,\mathbf{R}E,H)$. It defines a path structure. It is strict with the left translation of θ. The tautological forms are θ, $\theta^1=a^3\alpha^2$ and $\theta^2=a^{-3}\alpha^1$.

We can now compare with the previous proposition and the structural equations of the strict path geometry. That is to say, we compare the two sets of equations:

$$\begin{cases}\mathrm{d}\theta+\alpha^1\wedge\alpha^2=0\\ \mathrm{d}\alpha^1-2\alpha^1\wedge\theta=0\\ \mathrm{d}\alpha^2-2\theta\wedge\alpha^2=0\end{cases}\text{ and }\begin{cases}\mathrm{d}\theta+\theta^2\wedge\theta^1=0\\ \mathrm{d}\theta^2+3w\wedge\theta^2=\theta\wedge\tau^2\\ \mathrm{d}\theta^1-3w\wedge\theta^1=\theta\wedge\tau^1.\end{cases} \tag{241}$$

We read those equations in the section $(\alpha^1,\alpha^2,\theta)$. The first equation of both systems is indeed verified:

$$\mathrm{d}\theta+\theta^2\wedge\theta^1=\mathrm{d}\theta+\alpha^1\wedge\alpha^2=0. \tag{242}$$

The equations in second position:

$$\mathrm{d}\alpha^1-2\alpha^1\wedge\theta=0\text{ and }\mathrm{d}\theta^2+3w\wedge\theta^2=\theta\wedge\tau^2 \tag{243}$$

show that $\tau^2=0$ and w must be $\frac{2}{3}\theta$ along the section $(\alpha^1,\alpha^2,\theta)$. The last equations shows that $\tau^1=0$ and w is again $\frac{2}{3}\theta$.

As a consequence, the strict path structure on $\mathrm{SL}(2,\mathbf{R})$ has curvature:

$$\Omega=\begin{pmatrix}\frac{2}{3}\theta^2\wedge\theta^1&0&0\\0&-\frac{2}{3}\theta^2\wedge\theta^1&0\\0&0&\frac{2}{3}\theta^2\wedge\theta^1\end{pmatrix}. \tag{244}$$

One can think of $\mathrm{SL}(2,\mathbf{R})$ with the above strict path structure as a constant curvature model. Observe that one can vary the curvature by choosing different multiples of H. The curvature sign corresponds then to different choices of orientation.

The automorphism group of this structure is $\mathrm{SL}(2,\mathbf{R})\times\mathbf{R}^*$. The action is through left translations by $\mathrm{SL}(2,\mathbf{R})$ and right translations by $\mathbf{R}^*$ identified to the one parameter subgroup

$$\left\{\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}\,\middle|\,t\in\mathbf{R}\right\} \tag{245}$$

Indeed, this group acts simply transitively on the adapted coframe bundle P over $\mathrm{SL}(2,\mathbf{R})$ and preserves the connection.

IV.4 Isomorphisms of Cartan geometries

Definition IV.13 *Let $P_1 \to M_1$ and $P_2 \to M_2$ be two Cartan geometries modeled on a same $(\mathfrak{g}, \mathfrak{h})$ with respective connections ω_1 and ω_2. A geometric isomorphism is a diffeomorphism $f\colon M_1 \to M_2$ covered by a H-equivariant map $F\colon P_1 \to P_2$ such that $\omega_1 = F^*\omega_2$.*

Proposition IV.14 *A geometric isomorphism $f\colon M_1 \to M_2$ between effective Cartan geometries modeled on $(\mathfrak{g}, \mathfrak{h})$ determines a unique H-equivariant lift $F\colon P_1 \to P_2$ such that $F^*\omega_2 = \omega_1$.*

Proof We show the statement by assuming that $f\colon M_1 \to M_1$ is the identity map. Otherwise, we could compose f with f^{-1} and take as lift a first lift F composed with a second lift G^{-1}.

Let $F\colon M_1 \to M_1$ be lifting the identity and assume that $F^*\omega_1 = \omega_1$. We must have $F(p) = R_{\psi(p)}(p)$ since it lifts the identity. Hence

$$\omega_1 = F^*\omega_1 = R_\psi^*\omega_1 \tag{246}$$

$$= \psi^*\theta_H + \mathrm{Ad}(\psi)^{-1}\omega_1 \tag{247}$$

$$\iff \mathrm{Ad}(\psi)^{-1}\omega_1 - \omega_1 = -\psi^*\theta_H. \tag{248}$$

We show that this implies that ψ has values in a normal subgroup of G. It will imply $\psi \cong e$ since the space is effective. We do this by induction, following proposition II.23 (p. 13).

It is clear that $\psi(P_1) \subset H = N_0$. Hence we suppose that $\psi(P_1) \subset N_i$. Since $-\psi^*\theta_H(v) = \theta_H(\psi_* v) \in \mathfrak{n}_i$, we have $\psi(P_1) \subset N_{i+1}$. ▧

IV.5 Bianchi identities

The derivative of the curvature gives a Bianchi identities.

Lemma IV.15 *Let $P \to M$ be a Cartan geometry and ω_P its connection. We have*

$$\mathrm{d}\Omega = [\Omega \wedge \omega_P]. \tag{249}$$

Proof We differentiate by definition of the curvature.

$$\mathrm{d}\Omega = \mathrm{d}\left(\mathrm{d}\omega_P + \frac{1}{2}[\omega_P \wedge \omega_P]\right) \tag{250}$$

$$= \frac{1}{2}\mathrm{d}[\omega_P \wedge \omega_P] \tag{251}$$

$$= \frac{1}{2}([\mathrm{d}\omega_P \wedge \omega_P] - [\omega_P \wedge \mathrm{d}\omega_P]) \tag{252}$$

$$= [\mathrm{d}\omega_P \wedge \omega_P] \tag{253}$$

$$= \left[\left(\Omega - \frac{1}{2}[\omega_P \wedge \omega_P]\right) \wedge \omega_P\right] \tag{254}$$

$$= [\Omega \wedge \omega_P] - \frac{1}{2}[[\omega_P \wedge \omega_P] \wedge \omega_P] \tag{255}$$

$$= [\Omega \wedge \omega_P] \tag{256}$$

Indeed, $[[\omega_P \wedge \omega_P] \wedge \omega_P] = 0$ by using the Jacobi identity. ▧

In Riemannian geometry This Bianchi identity gives the two usual Bianchi identities. Indeed, with the (Levi-Civita) Cartan connection constructed above,

$$\omega_P = \begin{pmatrix} \omega & \theta \\ 0 & 0 \end{pmatrix}, \tag{257}$$

we have

$$\Omega = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \tag{258}$$

and hence

$$\mathrm{d}\Omega = \begin{pmatrix} \mathrm{d}W & 0 \\ 0 & 0 \end{pmatrix} = [\Omega \wedge \omega_P] = \begin{pmatrix} W \wedge \omega & W\theta \\ 0 & 0 \end{pmatrix}. \tag{259}$$

On the $\mathbf{R}^n$ factor we retrieve the first Bianchi identity:

$$W\theta = [W \wedge \theta] = 0 \tag{260}$$

and on the $\mathfrak{o}(n)$ factor we get the second:

$$\mathrm{d}W = W \wedge \omega. \tag{261}$$

IV.6 Mutations

Definition IV.16 *Let $P \to M$ be a Cartan geometry modeled on $(\mathfrak{g}, \mathfrak{h})$. Its torsion is the projection of the curvature Ω by $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.*

Note When the model space is reductive, that is to say there exists $\mathfrak{p}$ that is $\mathrm{Ad}(h)$-invariant and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, then the torsion is the $\mathfrak{p}$ factor of Ω.

Example In Riemannian geometry the torsion vanishes exactly for the Levi-Civita connection. It is indeed what we constructed by asking $\mathrm{d}\theta + [\omega \wedge \theta] = 0$.

Definition IV.17 *Let $(\mathfrak{g}_1, \mathfrak{h})$ and $(\mathfrak{g}_2, \mathfrak{h})$ be two geometric pairs sharing a same group H corresponding to $\mathfrak{h}$ and having two respective adjoint representations $\mathrm{Ad}_1\colon H \to \mathrm{Aut}(\mathfrak{g}_1)$ and $\mathrm{Ad}_2\colon H \to \mathrm{Aut}(\mathfrak{g}_2)$.*

A mutation is a linear isomorphism

$$\lambda\colon \mathfrak{g}_1 \to \mathfrak{g}_2 \tag{262}$$

such that

(1) for all $h \in H$ and $u \in \mathfrak{g}_1$, $\lambda(\mathrm{Ad}_1(h)(u)) = \mathrm{Ad}_2(h)(\lambda(u))$;

(2) $\lambda|_{\mathfrak{h}}$ is the identity;

(3) in $\mathfrak{g}/\mathfrak{h}$, we have $\lambda([u, v]) = [\lambda(u), \lambda(v)]$.

Examples The three constant curvature models for the Riemannian geometry are mutations. Let $\mathbf{R}^n = \mathrm{Eucl}_+(n)/\mathrm{SO}(n)$, $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $H^n_{\mathbf{R}} = \mathrm{SO}(n,1)/\mathrm{SO}(n)$. The three Lie algebras $\mathfrak{so}(n+1), \mathfrak{so}(n,1)$ and $\mathfrak{eucl}(n)$ are decomposed into $\mathfrak{so}(n) \oplus \mathbf{R}^n$. Note that $\mathfrak{so}(n)$ is the Lie algebra of a shared isotropy $H = \mathrm{SO}(n)$. Let $A \in \mathfrak{so}(n)$ and $v \in \mathbf{R}^n$. Then the mutations are deduced from the three following representations.

$$\mathfrak{eucl}(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \right\} \tag{263}$$

$$\mathfrak{so}(n+1) = \left\{ \begin{pmatrix} A & v \\ -{}^t v & 0 \end{pmatrix} \right\} \tag{264}$$

$$\mathfrak{so}(n,1) = \left\{ \begin{pmatrix} A & v \\ {}^t v & 0 \end{pmatrix} \right\} \tag{265}$$

Proposition IV.18 *Let $(\mathfrak{g}_1, \mathfrak{h})$ and $(\mathfrak{g}_2, \mathfrak{h})$ be two geometric pairs with a mutation $\lambda\colon \mathfrak{g}_1 \to \mathfrak{g}_2$.*

If $P \to M$ is a Cartan geometry modeled on $(\mathfrak{g}_1, \mathfrak{h})$ with Cartan connection ω_1 then

$$\omega_2 = \lambda \circ \omega_1 \tag{266}$$

gives a Cartan connection for $P \to M$ modeled on $(\mathfrak{g}_2, \mathfrak{h})$. Furthermore, the curvature Ω_1 becomes:

$$\Omega_2 = \lambda \circ \Omega_1 + \frac{1}{2}\left([\omega_2 \wedge \omega_2] - \lambda[\omega_1 \wedge \omega_1]\right). \tag{267}$$

Proof Since λ is an isomorphism, ω_2 is a linear isomorphism at each point: $\mathrm{T}_p P \to \mathfrak{g}_2$. It verifies the other properties since on the equivalent property we have:

$$\begin{aligned} R_\psi^* \omega_2 &= (\lambda \circ \omega_1)(R_{\psi_*}) && (268)\\ &= \lambda\left(\psi^* \theta_H + \mathrm{Ad}_1(\psi)^{-1} \omega_1\right) && (269)\\ &= \psi^* \theta_H + \mathrm{Ad}_2(\psi)^{-1} \lambda \circ \omega_1 && (270)\\ &= \psi^* \theta_H + \mathrm{Ad}_2(\psi)^{-1} \omega_2. && (271) \end{aligned}$$

The identity on Ω_2 follows by definition. ∎

Note If Ω_1 has vanishing torsion then Ω_2 does too since λ preserves $\mathfrak{h}$.

Definition IV.19 *Let $P \to M$ be a Cartan geometry with connection ω_P and curvature Ω. We define its curvature function*

$$K\colon P \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{272}$$

by

$$K(u,v)|_p = \Omega\left(\omega_P^{-1}(u)|_p, \omega_P^{-1}(v)|_p\right). \tag{273}$$

In fact, since Ω vanishes on $\mathfrak{h}$, we have a factorisation:

$$K\colon P \times \mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}. \tag{274}$$

Note The curvature function K has values in $\mathfrak{h}$ if, and only if, Ω has vanishing torsion.

Lemma IV.20 *If Ω has vanishing torsion then $K(K(u,v),w) = 0$.*

Definition IV.21 *Let $P \to M$ be a Cartan geometry with connection ω_P and curvature Ω. It has constant curvature if K does not depend on P.*

Theorem IV.22 *Let $P \to M$ be a Cartan geometry modeled on $(\mathfrak{g}_1, \mathfrak{h})$ with connection ω_P and curvature Ω. Assume it has constant curvature and vanishing torsion. Then $\lambda\colon \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathfrak{g}_1$ defined linearly by* id *($\mathfrak{g}_2$ is a linear copy of $\mathfrak{g}_1$) but with bracket*

$$[u,v]_{\mathfrak{g}_2} = [u,v]_{\mathfrak{g}_1} - K(u,v) \tag{275}$$

defines a mutant geometry on which $P \to M$ is flat.

Proof We prove first that $\mathfrak{g}_2$ is well defined. It only depends on wether the bracket is indeed a bracket of Lie algebra. It is certainly anti-symmetric and bilinear since K is a 2-form. The Jacobi identity is comes from the following computation.

$$\begin{aligned} [u,[v,w]_{\mathfrak{g}_2}]_{\mathfrak{g}_2} &= [u,[v,w]_{\mathfrak{g}_2}]_{\mathfrak{g}_1} - K(u,[v,w]_{\mathfrak{g}_2}) && (276)\\ &= [u,[v,w]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} - [u,K(v,w)]_{\mathfrak{g}_1} - K(u,[v,w]_{\mathfrak{g}_1}) + K(u,K(v,w)) && (277)\\ &= [u,[v,w]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} - [u,K(v,w)]_{\mathfrak{g}_1} - K(u,[v,w]_{\mathfrak{g}_1}) && (278)\\ -[v,[u,w]_{\mathfrak{g}_2}]_{\mathfrak{g}_2} &= -[v,[u,w]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} + [v,K(u,w)]_{\mathfrak{g}_1} + K(v,[u,w]_{\mathfrak{g}_1}) && (279)\\ [[u,v]_{\mathfrak{g}_2},w]_{\mathfrak{g}_2} &= \left[[u,v]_{\mathfrak{g}_1},w\right]_{\mathfrak{g}_1} - K([u,v]_{\mathfrak{g}_1},w) - [K(u,v),w]_{\mathfrak{g}_1} && (280) \end{aligned}$$

Hence the Jacobi identity only depends on a circular identity of $K(u,[v,w])$ and $[K(u,v),w]$. For this we use the Bianchi identity.

Let U,V,W be $\omega_P^{-1}(u),\omega_P^{-1}(v),\omega_P^{-1}(w)$. Then, since the curvature is constant, (we now take every bracket in $\mathfrak{g}_1$)

$$\mathrm{d}\Omega(U,V,W) = -\Omega([U,V],W) + \Omega(U,[V,W]) - \Omega(V,[U,W]) \tag{281}$$

and the Bianchi identity say this is equal to

$$[\Omega(U,V),\omega_P(W)] - [\omega_P(U),\Omega(V,W)] + [\omega_P(V),\Omega(U,W)]. \tag{282}$$

With a torsion free curvature we have also:

$$\Omega([U,V],W) = K([u,v] - K(u,v),w) = K([u,v],w). \tag{283}$$

So the Bianchi identity states:

$$-K([u,v],w) + K(u,[v,w]) - K(v,[u,w]) = [K(u,v),w] - [u,K(v,w)] + [v,K(u,w)] \tag{284}$$

finishing to prove that $[\cdot,\cdot]_{\mathfrak{g}_2}$ is a bracket.

Now we prove that we have indeed a mutation. We need to prove that $[\mathrm{Ad}(h)u,\mathrm{Ad}(h)v]_{\mathfrak{g}_2} = \mathrm{Ad}(h)[u,v]_{\mathfrak{g}_1}$. This equality will be proved if we show

$$K(\mathrm{Ad}(h)u,\mathrm{Ad}(h)v) = \mathrm{Ad}(h)K(u,v). \tag{285}$$

It is true since by constant curvature:

$$\Omega(U,V) = \mathrm{Ad}(h)^{-1}\Omega(\mathrm{Ad}(h)U,\mathrm{Ad}(h)V). \tag{286}$$

Finally, the new connection is indeed flat by the preceding proposition and a straightforward computation. ▧

IV.7 Covariant derivative

On a reductive space G/H with an $\mathrm{Ad}(H)$-invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{287}$$

one can decompose any connection ω_P along this decomposition:

$$\omega_P = \omega_{\mathfrak{h}} + \omega_{\mathfrak{p}}. \tag{288}$$

Proposition IV.23 *Let $P \to M$ be a Cartan geometry modeled on a reductive pair $(\mathfrak{g},\mathfrak{h})$ with connection ω_P. Then one can define the operator ∇^ω on vector fields:*

$$\nabla^\omega_X(Y) = X(\omega_{\mathfrak{p}}(Y)) + \left[\omega_{\mathfrak{h}}(X),\omega_{\mathfrak{p}}(Y)\right] \tag{289}$$

and it defines a covariant derivative with values in $\mathfrak{g}$.

Proof It is clear that ∇^ω is bilinear. If f is smooth function then $\nabla^\omega_{fX}(Y) = f\nabla^\omega_X(Y)$ and $\nabla^\omega_X(fY) = \mathrm{d}f(X) + f\nabla^\omega_X(Y)$ showing that it is a covariant derivation. ▧

Note In the case of an affine connection ($\mathfrak{p} = \mathbf{R}^n$ is commutative and $\mathfrak{h} \subset \mathfrak{gl}(n)$) then it can be shown by hand that the usual torsion and curvature of ∇^ω corresponds to the $\mathfrak{p}$ factor and $\mathfrak{h}$ factor of Ω respectively.

V — Cartan's method and the equivalence problem

One of the basic problems in geometry is to understand the equivalence between geometric objects. For instance, given two Riemannian manifolds when are they locally or globally isometric? The main idea of Cartan's method is to associate to a manifold with a geometric structure another manifold (with higher dimension) where the geometric structure is given by a parallelism of its cotangent bundle. The parallelism can be defined in very general situations but when the geometric structure is simple enough one can describe it by an important mathematical object: Cartan connections on principal bundles.

V.1 Differential ideals and the equivalence problem

We will work in the C^∞ category. Let Let M be an n-dimensional manifold and $\Omega^*(M)$ be the set of sections of the space $\Lambda T^* M$, the graded algebra of the exterior powers of the cotangent bundle. The space $\Omega^*(M)$ is the space of all the differential forms of M.

Definition V.1 *A differential ideal $I \subset \Omega^*(M)$ is an ideal for the exterior algebra and closed under exterior derivative.*

Definition V.2 *If I is a differential ideal, an integral submanifold is an immersion $\phi\colon N \to M$ such that $\phi^*\omega = 0$ for any $\omega \in I$.*

Note Note that if α is a 1-form that annihilates a distribution then since

$$\mathrm{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{290}$$

the ideal generated by such α's is closed under the exterior derivative if, and only if, $[X,Y]$ belongs also to the distribution.

So the most natural example arises from the ideal I_D of forms which annihilate a distribution D. In that case the ideal is a differential ideal if, and only, if the distribution is involutive and Frobenius theorem is stated in this language as the following.[2]

Theorem V.3 (Frobenius) *Let I be a differential ideal locally generated by $(n-p)$ independent 1-forms. Then, for each $x \in M$, there exists a unique maximal (of dimension p) connected integral manifold of I passing through x.*

In fact, it suffices that the 1-forms in the statement be of regularity C^1.

Example 1 If the ideal is generated by a single 1-form θ, then being a differential ideal means that $\mathrm{d}\theta = \theta\wedge\omega$, for ω a 1-form. (Hence $\mathrm{d}\theta\wedge\theta = 0$.) The extreme opposite would be that θ satisfies $\mathrm{d}\theta\wedge\theta \neq 0$ at every point. (It is the contact hypothesis.)

Example 2 If the ideal is generated by the 1-form $\mathrm{d}y - p\,\mathrm{d}x$ and $\mathrm{d}p - F(x,y,p)\,\mathrm{d}x$ in $\mathbf{R}^3$ we obtain one dimensional integral submanifolds which correspond to solutions of a second order differential equation.

The equivalence problem in its simplest form is the following. Let M_1 and M_2 be manifolds of the same dimension n and $\{\omega_1^i\}$ and $\{\omega_2^i\}$ be coframe sections, that is, n independent 1-forms. Is there a diffeomorphism

$$\psi\colon M_1 \to M_2 \text{ such that } \psi^*\omega_2^i = \omega_1^i\ ? \tag{291}$$

To answer to that question Cartan used the graph method. The idea is to find the map ψ by its graph in $M_1 \times M_2$. On the other hand, the graph is obtained as an integral submanifold of a differential ideal.

Theorem V.4 *Let M_1 and M_2 be manifolds and π_1, π_2 the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. Let $(\omega_2^i)_{1\le i\le n}$ be a basis of 1-forms of M_2 and $(\omega_1^i)_{1\le i\le n}$ be a family of forms M_1 respectively. If the ideal of forms on $M_1 \times M_2$ generated by*

$$\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i) \tag{292}$$

is a differential ideal then, for each pair $(x,y) \in M_1 \times M_2$, there exists a a map $\phi : U \to M_2$ such that $\phi(x) = y$ and

$$\phi^*(\omega_2^i) = \omega_1^i. \tag{293}$$

[2] See F. Warner, *Foundations of differentiable manifolds and Lie groups.*

Proof The generating 1-forms are linearly independent because ω_2^i are linearly independent. By Frobenius theorem, there exists a unique maximal submanifold G of dimension n_1 containing a point $(x, y) \in M_1 \times M_2$ which is an integral submanifold of the differential ideal.

We show now that the submanifold is locally a graph. Consider a vector $(v_1, v_2) \in \mathrm{T}G \subset \mathrm{T}M_1 \times \mathrm{T}M_2$. If $(\pi_1)_*(v_1, v_2) = 0$ then $v_1 = 0$ and therefore $\pi_1^*(\omega_1^i)(v_1, v_2) = 0$ which implies (because G is an integral submanifold of the ideal) that $\pi_2^*\omega_2^i(v_1, v_2) = 0$. We conclude that $v_2 = 0$. Therefore $\mathrm{T}_{(x,y)}G$ is isomorphic to $\mathrm{T}_{m_1}M_1$ and π_1 is a local diffeomorphism.

Let $F\colon U \to G$ be a local inverse of π_1. We have that $F(m) = (m, \phi(m))$ for a certain function $\phi\colon U \to M_2$. Moreover, as $\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i) = 0$ on G, we obtain $F^*(\pi_1^*(\omega_1^i) - \pi_2^*(\omega_2^i)) = 0$ and therefore $\omega_1^i = \phi^*(\omega_2^i)$. ▧

With constant structures One special case occurs if we suppose that the coframes in M_1 and M_2 verify

$$\mathrm{d}\omega^i = c^i_{jk}\omega^j \wedge \omega^k, \tag{294}$$

with c^i_{jk} constant numbers shared by both M_1 and M_2. Then, observe that

$$\begin{aligned}
\mathrm{d}\left(\pi_1^*\omega_1^i - \pi_2^*\omega_2^i\right) &= \pi_1^*(\mathrm{d}\omega_1^i) - \pi_2^*(\mathrm{d}\omega_2^i) && (295)\\
&= \pi_1^*\left(c^i_{jk}\omega_1^j \wedge \omega_1^k\right) - \pi_2^*\left(c^i_{jk}\omega_2^j \wedge \omega_2^k\right) && (296)\\
&= c^i_{jk}\left(\pi_1^*(\omega_1^j \wedge \omega_1^k) - \pi_2^*(\omega_2^j \wedge \omega_2^k)\right) && (297)\\
&= c^i_{jk}\left(\left(\pi_1^*\omega_1^j - \pi_2^*\omega_2^j\right) \wedge \pi_1^*\omega_1^k - \pi_2^*\omega_2^j \wedge \left(\pi_2^*\omega_2^k - \pi_1^*\omega_1^k\right)\right) && (298)
\end{aligned}$$

so that the ideal is differential and M_1 and M_2 are hence locally equivalent.

The case of Lie groups is particularly important. With any left-invariant frame (X_i) and its coframe (ω^i) we get structure constants c^i_{jk} verifying the preceding condition:

$$\mathrm{d}\omega^i = c^i_{jk}\omega^j \wedge \omega^k. \tag{299}$$

A basis of 1-forms (ω^i) on a manifold M is called a parallelism of M. An automorphism of a parallelism (ω^i) defined over a manifold M is a diffeomorphism $\phi\colon M \to M$ such that $\phi^*\omega^i = \omega^i$. From unicity in the theorem above we obtain the following corollary.

Corollary V.5 *Any automorphism of a parallelism with a fixed point is the identity.*

In particular the dimension of the group of automorphisms is at most the dimension of the manifold. This gives a way to compute the maximal dimension of the automorphism group of a geometry. The idea is to construct, from the geometric data, another manifold with a canonical parallelism. The dimension of that manifold gives the dimension of the group of automorphisms. With Cartan geometries, this canonical parallelism is a Cartan connexion.

Theorem V.6 *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let M be a manifold with a $\mathfrak{g}$-valued 1-form ϕ defined on $\mathrm{T}M$ satisfying the structure equation*

$$\mathrm{d}\phi + \frac{1}{2}[\phi \wedge \phi] = 0. \tag{300}$$

Then for any $m \in M$ there exists a map $f\colon U \to G$ defined on a neighborhood of m such that $\phi = f^*\theta$ where θ is the Maurer-Cartan form of G. Moreover if $g\colon U \to G$ is another map satisfying this condition then $g = L_h \circ f$ for a certain constant $h \in G$.

Proof We consider, in the product $M \times G$, the Lie algebra valued form

$$\omega = \pi_1^* \phi - \pi_2^* \theta, \tag{301}$$

where π_1 and π_2 are the projections of the product on each factor.

Let I be the ideal generated by the components ω^i_j of ω. This is a differential ideal because

$$\begin{aligned} d\omega &= \pi_1^* \, d\phi - \pi_2^* \, d\theta && (302)\\ &= -\frac{1}{2}\pi_1^* [\phi \wedge \phi] + \frac{1}{2}\pi_2^* [\theta \wedge \theta] && (303)\\ &= -\frac{1}{2}\left([\pi_1^*\phi - \pi_2^*\theta \wedge \pi_1^*\phi] + [\pi_2^*\theta \wedge \pi_1^*\phi] - [\pi_2^*\theta \wedge \pi_2^*\theta]\right) && (304)\\ &= -\frac{1}{2}\left([\omega \wedge \pi_1^*\phi] + [\pi_2^*\theta \wedge \omega]\right) && (305)\\ &= -\frac{1}{2}[\omega \wedge \omega]. && (306) \end{aligned}$$

We invoke the previous theorem to conclude the existence of the map $f\colon U \to G$. A submanifold passing through another point $(m_0, hf(m_0))$ is clearly given by $(m, hf(m))$ and by unicity this implies that $g = L_h \circ f$. ▧

V.2 Developing map and Flat Cartan geometries

V.2.1 Path development

Lemma V.7 *Let $f\colon [0,1] \to \mathfrak{g}$ be a smooth function. Let $\omega\colon TX \to \mathfrak{g}$ be a complete parallelism (its constant vector fields $\omega^{-1}(v)$ are complete, i.e. have flows fully defined on* **R***) verifying the structural equation. Then the differential equation*

$$\gamma^* \omega = f \, dt \tag{307}$$

has a solution $\gamma\colon [0,1] \to X$ that is unique once an initial condition $\gamma(0) = x \in X$ is given.

Proof Note that $f \, dt$ verifies the structural equation. By Cartan's method, a local solution does always exist and is unique once an initial condition is given. We have to show that a solution can always be extended to the full interval $[0,1]$.

Suppose that a local solution γ is only defined for $t < 1$. Then $\gamma(t)$ escapes every compact set of X when $t \to 1$. But when $t \to 1$, $f(t) \to v \in \mathfrak{g}$ and a global solution to $\gamma^*\omega = v$ exists by completeness of ω on X. A contradiction. ▧

The development of paths follows from this lemma. We let $\omega = \theta_G$ be the Maurer-Cartan form of a Lie group G. Any path $\delta\colon [0,1] \to P$ defined on a manifold P equipped with a $\mathfrak{g}$-valued 1-form $\omega\colon TP \to \mathfrak{g}$ gives by pulling back the 1-form $\delta^*\omega$. Then by what precedes, $\delta^*\omega = \gamma^*\theta_G$ for a path $\gamma\colon [0,1] \to G$.

In our context P will be the total space of a principal bundle $P \to M$ and the form ω will be a Cartan connection.

Definition V.8 *Let P be a smooth manifold equipped with a $\mathfrak{g}$-valued 1-form ω. Any path $\delta\colon [0,1] \to P$ determines a path $D_\omega(\delta)\colon [0,1] \to G$ such that*

$$\delta^*\omega = D_\omega(\delta)^*\theta_G \tag{308}$$

and $D_\omega(\delta)$ is unique as soon as $D_\omega(\delta)(0) \in G$ is prescribed.

The map giving the endpoint:

$$E_\omega(\delta) = D_\omega(\delta)(0)^{-1} D_\omega(\delta)(1) \tag{309}$$

is well defined and does not depend on the choice of $D_\omega(\delta)$.

The map E_ω is defined on the space of the paths of P. Its values are in G. Now, the goal is to obtain a map

$$F_\omega \colon P \to G \tag{310}$$

that would be a complete integration of ω:

$$F_\omega^* \theta_G = \omega. \tag{311}$$

The most natural way would be to fix $p \in P$ and define $F_\omega(z)$ as $E_\omega(\delta)$ for any path δ joining p to z. With this goal in mind, one needs to compare the different values of E_ω for different paths joining the same points.

A natural assumption is to compare paths that have the same homotopy class in $\pi_1(P, p)$. Those have indeed same endpoints by E_ω if the space P is flat.

Lemma V.9 *If there exists F_ω such that $F_\omega^* \theta_G = \omega$ then ω verifies the structural equation*

$$\mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] = 0. \tag{312}$$

Proof This follows by naturality of the pulling-back and the fact that θ_G itself verifies the structural equation. ▨

Since η is defined on the whole $\mathrm{T}[0,1]$, the function f is bounded.

Proposition V.10 *Let P be a smooth manifold equipped with a $\mathfrak{g}$-valued 1-form ω that verifies the structural equation. If $H \colon [0,1] \times [0,1] \to P$ is an homotopy between $\delta_1 = H(0,t)$ and $\delta_2 = H(1,t)$ then $E_\omega(\delta_1) = E_\omega(\delta_2)$.*

Proof Since ω verifies the structural equation, one can apply Cartan's method. Again, by completeness of the Maurer-Cartan form, it defines a complete integral $H_G \colon [0,1] \times [0,1] \to G$ such that

$$H_G^* \theta_G = H^* \omega. \tag{313}$$

Since H is a homotopy, $H^* \omega$ vanishes on $[0,1] \times \{0,1\}$. Hence H_G does too and it furnishes an homotopy in G. Therefore H_G has two equal endpoints for $H_G(0,t) = D_\omega(\delta_1)$ and $H_G(1,t) = D_\omega(\delta_2)$. ▨

Definition V.11 *Let P be a smooth manifold equipped with a $\mathfrak{g}$-valued 1-form ω verifying the structural equation. The monodromy morphism*

$$\Phi_\omega \colon \pi_1(P, p) \to G \tag{314}$$

is the value of $E_\omega(\delta)$ for any δ realizing a chosen class $[\delta] \in \pi_1(P, p)$. It is a group homomorphism by concatenation of paths. Its image is the monodromy subgroup $\Phi_\omega(\pi_1(P, p)) \subset G$.

Corollary V.12 *Let P be a smooth manifold equipped with a $\mathfrak{g}$-valued 1-form. There exists a global map $F_\omega \colon P \to G$ such that*

$$F_\omega^* \theta_G = \omega \tag{315}$$

if, and only if, ω verifies the structural equation and its monodromy is trivial.

V.2.2 Flat Cartan geometries

Now we consider a Cartan geometry $P \to M$. The development F_ω of P in G will allow to define a developing map from $\widetilde{M}$ to G/H. Here we can see $\widetilde{M}$ as the space of the paths of M modulo homotopy.

A first step is to verify that the principal H-bundle structure on P is compatible with the one on G under F_ω.

Lemma V.13 *Let $P \to M$ be a Cartan geometry modeled on a reductive pair $(\mathfrak{g}, \mathfrak{h})$ with a (non-necessarily flat) Cartan connection ω. Let $\delta\colon [0,1] \to P$ be a path and $\psi\colon [0,1] \to H$ be a smooth function. Then*

$$D_\omega(\delta\psi) = D_\omega(\delta)\psi \tag{316}$$

if we have the compatibility $D_\omega(\delta\psi)(0) = D_\omega(\delta)(0)\psi(0)$.

Proof Both $D_\omega(\delta\psi)$ and $D_\omega(\delta)\psi$ are paths on G with same initial point. We only need to check that their derivatives are equal since the unicity of the development of paths would conclude. Indeed we have:

$$D_\omega(\delta\psi)^*\theta_G = (\delta\psi)^*\omega \tag{317}$$

$$= \mathrm{Ad}(\psi)^{-1}\delta^*\omega + \psi^*\theta_H \tag{318}$$

$$= \mathrm{Ad}(\psi)^{-1}D_\omega(\delta)^*\theta_G + \psi^*\theta_H \tag{319}$$

$$= (D_\omega(\delta)\psi)^*\theta_G. \tag{320}$$

Proposition V.14 *Let $P \to M$ be a Cartan geometry modeled on an effective Kleinian pair $(\mathfrak{g}, \mathfrak{h})$ with a flat Cartan connection ω_P. Then there exists a local diffeomorphism*

$$D\colon \widetilde{M} \to G/H \tag{321}$$

called a developing map.

Proof With the universal cover $\pi_1\colon \widetilde{M} \to M$ we define the pulled-back bundle $\widetilde{P}$ by

$$\widetilde{P} = \left\{(p,x) \in P \times \widetilde{M} \,\middle|\, \pi_P(p) = \pi_1(x)\right\}. \tag{322}$$

We have the projection maps $\widetilde{\pi_1}\colon \widetilde{P} \to P$ and $\pi_{\widetilde{P}}\colon \widetilde{P} \to \widetilde{M}$. The pulled-back Cartan connection $\omega_{\widetilde{P}} = \widetilde{\pi_1}^*\omega_P$ defined on $\widetilde{P}$ has still flat curvature by naturality.

$$\begin{array}{ccccccc}
\mathrm{T}\widetilde{P} & \xrightarrow{\widetilde{\pi_1}_*} & \mathrm{T}P & \xrightarrow{\omega_P} & \mathfrak{g} & \xleftarrow[\omega_G]{} & \mathrm{T}G \\
\downarrow & & \downarrow & & & & \downarrow \\
\widetilde{P} & \xrightarrow{\widetilde{\pi_1}} & P & & & & G \\
\downarrow{\scriptstyle \pi_{\widetilde{P}}} & & \downarrow{\scriptstyle \pi_P} & & & & \downarrow{\scriptstyle \pi_G} \\
\widetilde{M} & \xrightarrow{\pi_1} & M & & & & G/H
\end{array} \tag{323}$$

The short exact sequence of the fiber bundle $H \to \widetilde{P} \to \widetilde{M}$ shows that

$$\pi_1(H,e) \to \pi_1(\widetilde{P},p) \to \pi_1(\widetilde{M},x) = \{e\}. \tag{324}$$

By composition with the monodromy morphism, we obtain the exact sequence:

$$\{e\} = \Phi_{\omega_{\widetilde{P}}}(\pi_1(H,e)) \to \Phi_{\omega_{\widetilde{P}}}(\pi_1(\widetilde{P},p)) \to \{e\} \tag{325}$$

showing that the monodromy of $\widetilde{P}$ is trivial. (Note that $\Phi_{\omega_{\widetilde{P}}}(\pi_1(H,e))$ is trivial since $H \subset \widetilde{P}$ is developed by the identity diffeomorphism to $H \subset G$.)

By the preceding corollary, we obtain a development

$$F_{\omega_{\widetilde{P}}}\colon \widetilde{P} \to G. \tag{326}$$

It is necessarily a local diffeomorphism that preserves the fibers since $\omega_{\widetilde{P}}$ identifies the tangent space of each fiber with $\mathfrak{h}$.

Therefore, $F_{\omega_{\widetilde{P}}}$ descends to a developing map

$$D\colon \widetilde{M} \to G/H \tag{327}$$

that is again a local diffeomorphism.

PROPOSITION V.15 *Under the same assumptions, the developing map* $D\colon \widetilde{M} \to {}^{G}\!/_{H}$ *is paired with a holonomy morphism*

$$\rho\colon \pi_1(M,x) \to G \tag{328}$$

that is equivariant:

$$\forall \gamma \in \pi_1(M,x), \forall y \in \widetilde{M},\ D(\gamma y) = \rho(\gamma)D(y). \tag{329}$$

PROOF Recall that with the universal cover $\pi_1\colon \widetilde{M} \to M$ we constructed

$$\widetilde{P} = \left\{(x,p) \in \widetilde{M} \times P \,\middle|\, \pi_1(x) = \pi_P(p)\right\}. \tag{330}$$

The left action of $\pi_1(M,x)$ on $\widetilde{M}$ can be lifted to $\widetilde{P}$ by:

$$\forall \gamma \in \pi_1(M),\ \gamma\cdot(x,p) = (\gamma\cdot x, p). \tag{331}$$

Hence $\widetilde{\pi_1} \circ \gamma = \widetilde{\pi_1}$. We obtain:

$$\omega_{\widetilde{P}} = \widetilde{\pi_1}^* \omega_P = \gamma^* \widetilde{\pi_1}^* \omega_P = \gamma^* \omega_{\widetilde{P}}. \tag{332}$$

Since γ is an automorphism of $\widetilde{P}$, it corresponds to a left translation $\rho(\gamma)$ of G. For indeed, with any path η based at $p \in \widetilde{P}$, the forms $\eta^*\omega_{\widetilde{P}}$ and $\eta^*\gamma^*\omega_{\widetilde{P}}$ are equal and hence the endpoints of their developments differ by $\rho(\gamma)$ which does not depends on η. It can be checked that ρ is indeed a morphism by concatenation of loops in $\pi_1(M,x)$. It verifies the equivariance property by what precedes. ▨

THEOREM V.16 *Let* $P \to M$ *be a Cartan geometry modeled on an effective Kleinian pair* $(\mathfrak{g},\mathfrak{h})$ *with a flat Cartan connection* ω_P*. If the Cartan connection* ω_P *is complete, that is to say every* $\omega_P^{-1}(v)$ *vector field is complete (its flow is defined on* $\mathbf{R}$*), then the developing map*

$$D\colon \widetilde{M} \to {}^{G}\!/_{H} \tag{333}$$

is a covering map. If ${}^{G}\!/_{H}$ *is also simply connected then it follows, with* $\Gamma = \rho(\pi_1(M,x))$ *the image of the holonomy morphism, that* D *is a diffeomorphism and*

$$M \cong {}_{\Gamma}\backslash{}^{G}\!/_{H}. \tag{334}$$

PROOF The developing map D is a cover if, and only if, it has the lifting property. That is to say, we check that D can lift uniquely any path in ${}^{G}\!/_{H}$ with the choice of base points $x \in \widetilde{M}$ and $D(x) \in {}^{G}\!/_{H}$.

Any smooth path $\delta\colon [0,1] \to {}^{G}\!/_{H}$ can be lifted in G by a path $\widetilde{\delta}\colon [0,1] \to G$. Then $\widetilde{\delta}^*\theta_G = f\,\mathrm{d}t$. By lemma V.13 (p. 38), there exists a unique path $\gamma\colon [0,1] \to \widetilde{P}$ such that $\gamma^*\omega_{\widetilde{P}} = f\,\mathrm{d}t$. Then the projection of γ in $\widetilde{M}$ lifts δ by construction. ▨

COROLLARY V.17 *Let* P *be smooth manifold equipped with a complete parallelism* $\omega\colon \mathrm{T}P \to \mathfrak{g}$ *verifying the structural equation. If* P *is simply connected, then* P *is diffeomorphic to* G *the unique simply connected Lie group with Lie algebra* $\mathfrak{g}$*. Its group law is the concatenation of paths.*

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