Introduction to Cartan geometry

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I — Introduction

Cartan geometries are a solution to the very general question: *what is a geometric structure?* Riemannian geometry, conformal geometry and projective geometry are examples of geometric situations.

The mindset is the following. A Cartan geometry should first be a manifold with an homogenous space attached to each point. For instance in Riemannian geometry each point has an attached Euclidean space by equipping the tangent space with the Riemannian metric. This data is then equipped with a Cartan connection explaining how the homogeneous spaces are infinitesimally *connected*.

When one has two different Cartan geometries, one can ask if they are equivalent. For instance, when are two Riemannian manifold isometric or at least locally isometric? This is a deep question known under the general name of the *equivalence problem*. In Riemannian geometry, the differential system $g = \sum \mathrm{d}x_i^2$ asks wether the space is locally euclidean. It is the case if, and only if, a curvature tensor vanishes. Cartan geometries give a similar procedure for all the geometries: a curvature tensor vanishes if, and only if, the space is locally homogeneous.

But when the curvature is not zero, the equivalence problem is harder to solve. What is the meaning of two curvature on two different spaces being equal? Cartan's method for the equivalence problem is a general procedure to study and solve this problem in many situations. An important example is given by the class of the symmetric spaces: those are the Riemannian spaces that are not flat but have a parallel curvature tensor. With Cartan's method one can verify when two spaces with this property are locally equivalent or not.

In this course, we will describe Cartan geometries and introduce the local equivalence problem between geometric structures. The main global problem we will deal with is the classification of smooth Anosov flows on a compact three manifold and, more generally, of non-compact automorphisms groups acting on a compact manifold preserving a contact distribution and two transverse lines contained in the contact plane at each point of the manifold.

II — Lie groups and homogenous spaces

II.1 Lie groups and Lie algebras

We start with the definition of a Lie group. General references for this section are [War83; Kna02].

Definition II.1 *A Lie group is a group G that is also a differential manifold and such that the operations of multiplication and inverse are smooth. That is, the maps $G \times G \to G$ and $G \to G$ given by $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth.*

Definition II.2 *A homomorphism $H \to G$ of Lie groups is a group homomorphism which is a smooth map. The automorphism group of H is the group of bijective homomorphisms of H into H.*

Note that if we ignore continuity in the definition of homomorphisms of Lie groups one might obtain a much larger set.

To each Lie group is associated a Lie algebra which can be thought as the space of tangent vectors at the identity of the group.

Definition II.3 *A Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is a real vector space of finite dimension equipped with a bilinear map*

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}, \tag{1}$$

satisfying, for any $x, y, z \in \mathfrak{g}$ the anti-commutativity property $[x, y] = -[y, x]$ and the Jacobi identity:

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]. \tag{2}$$

Definition II.4 *A homomorphism $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ between Lie algebras is a homomorphism of vector spaces preserving the Lie bracket, that is, $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{h}$. The automorphism group of $\mathfrak{h}$ is the group of bijective homomorphisms of $\mathfrak{h}$ into $\mathfrak{h}$.*

Let G be a Lie group. If $a \in G$ is fixed, then one can consider the translations $L_a(g) = ag$ and $R_a(g) = ga$ called left and right multiplication respectively.

Definition II.5 *A vector field X on a Lie group G is* left invariant *if, for any $a \in G$, $(L_a)_*(X) = X$. Similarly, it is* right invariant *if $(R_a)_*(X) = X$.*

Note that this condition means $(L_a)_*(X(g)) = X(ag)$.

An important consequence of this definition is that left (or right) invariant vector fields are determined by their value at the identity of the group and the Lie bracket of two invariant vector fields is again invariant. Therefore the set of left invariant vector fields forms a Lie algebra that can be identified to the tangent space of the group at the identity.

Definition II.6 *The Lie algebra of a Lie group G is the set*

$$\mathfrak{g} = \left\{X \in C^\infty(\mathrm{T}G) \,\middle|\, \forall a \in G,\ (L_a)_*(X) = X\right\} \tag{3}$$

of left invariant vector fields on G equipped with the bilinear map given by the bracket between vector fields.

A subgroup $H \subset G$ which is a Lie group and such that the inclusion map is smooth is a called a Lie subgroup. Imposing that the inclusion is an embedding is equivalent to assuming that the subgroup is closed as a subspace of G (this result is called the closed-subgroup theorem or Cartan theorem).

The relation between Lie algebra homomorphisms and Lie group homomorphisms is described by the following.

Theorem II.7 *Let H and G be Lie groups and $\phi\colon H \to G$ a smooth homomorphism. Then $\mathrm{d}\phi_e\colon \mathfrak{h} \to \mathfrak{g}$ is a homomorphism. Conversely, if $\alpha\colon \mathfrak{h} \to \mathfrak{g}$ is a homomorphism and H is simply connected, then there exists a unique smooth homomorphism $\phi\colon H \to G$ such that $\alpha = \mathrm{d}\phi_e$.*

Examples

(1) The additive group $\mathbf{R}^n$. The automorphism group coincides with linear isomorphisms of $\mathbf{R}^n$, that is to say $\mathrm{GL}(n,\mathbf{R})$. But note that the full group of group automorphisms (not necessarily continuous) of the group $\mathbf{R}^n$ contains non-linear maps.

(2) The set of matrices with determinant one $\mathrm{SL}(n,\mathbf{R})$ and the usual product of matrices as group law.

(3) Let G be a Lie group, $N \subset G$ be a normal subgroup and $K \subset G$ a subgroup satisfying $N \cap K = \{e\}$ and $G = NK$. (This last condition means that $g \in G$ can always be written as nk with $n \in N$ and $k \in K$.) In this conditions, we say that G is the semidirect product of K and N and write $G = N \rtimes K$. Observe that if $g_1 = n_1 k_1$ and $g_2 = n_2 k_2$ then $g_1 g_2 = n_1 (k_1 n_2 k_1^{-1}) k_1 k_2$.

An example is given by the affine linear group $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n \rtimes \mathrm{GL}(n,\mathbf{R})$. Given an affine transformation T acting on the affine plane $\mathbf{R}^n$, the choice of a base point $0 \in \mathbf{R}^n$ allows to write

$$T(x) = c + f(x) \tag{4}$$

with $c \in \mathbf{R}^n$ and $f \in \mathrm{GL}(n,\mathbf{R})$. This decomposition is unique. Hence $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n\,\mathrm{GL}(n,\mathbf{R})$. Note that the change of the base point from $0 \in \mathbf{R}^n$ to $\zeta \in \mathbf{R}^n$ translates to:

$$\zeta + T(x - \zeta) = \zeta + (c - f(\zeta)) + f(x) \tag{5}$$

therefore the linear part f of T is independent from the choice of the base point, but the translational part depends on it.

The composition of two transformations T_1, T_2 is given by:

$$T_1(T_2(x)) = c_1 + f_1(c_2 + f_2(x)) = (c_1 + f_1(c_2)) + f_1 f_2(x) \tag{6}$$

and it proves that $\mathrm{Aff}(\mathbf{R}^n)$ is indeed the semidirect product $\mathbf{R}^n \rtimes \mathrm{GL}(n,\mathbf{R})$.

Note that a convenient representation of the affine group into $\mathrm{GL}(n+1,\mathbf{R})$ is given by

$$(c, f) \mapsto \begin{pmatrix} f & c \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Equivalently, semidirect products $G = N \rtimes K$ are in correspondance with split exact sequences

$$1 \to N \to G \to K \to 1 \tag{8}$$

and in the case of the affine group, we have indeed

$$0 \to \mathbf{R}^n \to \mathrm{Aff}(\mathbf{R}^n) \to \mathrm{GL}(n,\mathbf{R}) \to 1 \tag{9}$$

with the last morphism being independent of the choice of a base point and therefore is indeed restricted to the identity on $\mathrm{GL}(n,\mathbf{R})$.

(4) The three dimensional Heisenberg group Heis(3) is defined as

$$\mathrm{Heis}(3) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| (x, y, z) \in \mathbf{R}^3 \right\} \tag{10}$$

The group law is again the matrix product and is described by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+x\cdot y' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Another description of the same group is given by $\mathbf{C} \times \mathbf{R}$ with the (additive) group law

$$(x + \boldsymbol{i} y, z) \cdot (x' + \boldsymbol{i} y', z') = \left((x+x') + \boldsymbol{i}(y+y'), z + z' + \frac{1}{2}(xy' - yx') \right). \tag{12}$$

Both descriptions are compatible. One can start with the Lie algebra:

$$\mathfrak{heis}(3) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{13}$$

The exponential of an element is

$$\exp\left(\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

Therefore $\exp\colon \mathfrak{heis}(3) \to \mathrm{Heis}(3)$ is a diffeomorphism. The group law furnishes a law on the Lie algebra by taking the logarithm:

$$X \cdot Y = \log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] \tag{15}$$

and this law on $\mathfrak{heis}(3)$:

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x' & z' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x+x' & z+z'+\frac{1}{2}(xy'-yx') \\ 0 & 0 & y+y' \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

gives the second description.

The automorphism group of a simply connected Lie group coincides with the automorphism group of its Lie algebra. In the case of the Heisenberg group (which is diffeomorphic to $\mathbf{R}^3$) one can use the group operation on the Lie algebra to determine the automorphisms.

Proposition II.8 *The automorphism group of* Heis(3) *(described by coordinates* $(x + \boldsymbol{i} y, t) = (z, t) \in \mathbf{C} \times \mathbf{R}$*) is generated by the following transformations.*

(a) Transformations $(z, t) \mapsto (A(z), t)$ *where* $A\colon \mathbf{C} \to \mathbf{C}$ *is symplectic with respect to the form* $\mathrm{Im}(z\overline{z'}) = xy' - yx'$.

(b) Dilations $(z, t) \mapsto (az, a^2 t)$*, with* $a \in \mathbf{R}_+^*$.

(c) Conjugations by a translation $(a + \boldsymbol{i} b, c) \in \mathrm{Heis}(3)$*:* $(x + \boldsymbol{i} y, t) \mapsto (x + \boldsymbol{i} y, t + ay - bx)$.

(d) The inversion map $(z, t) \mapsto (\overline{z}, -t)$.

Proof We decompose an automorphism $\phi\colon \mathrm{Heis}(3) \to \mathrm{Heis}(3)$ by decomposing its derivative $\mathrm{d}\phi_e\colon \mathfrak{heis}(3) \to \mathfrak{heis}(3)$. With a linear automorphism $\mathrm{d}\phi_e$, we can write $\mathrm{d}\phi_e(x + \boldsymbol{i} y, t) = (A(x, y, t), at + bx + cy)$, with A a linear transformation and a, b, c three real numbers.

We note that an automorphism has to centralize the center of the group: if ζ is in the center, then $0 = \mathrm{d}\phi_e[\zeta, \cdot] = [\mathrm{d}\phi_e\zeta, \mathrm{d}\phi_e\cdot] = [\mathrm{d}\phi_e\zeta, \cdot]$. Therefore A can not depend on t. (The center of $\mathfrak{heis}(3)$ is exactly $(0, t)$.)

From $(A(x,y), at+bx+cy)$ one can compose by the conjugation by a translation such that $d\phi_e$ becomes $(A(x,y), at)$. (Choose the translation $(-c+\boldsymbol{i}b, 0)$.)

Next, if a is negative then we compose by an inversion. We obtain $(A'(x,y), |a|t)$ with A' that is either A or $\overline{A}$. Then we can compose by a dilatation by $\lambda = \sqrt{|a|^{-1}}$ so that we obtain $(\lambda A'(x,y), t)$.

Now, because t is fixed, $\lambda A'$ must be a symplectic transformation of $\mathbf{C}$. ▧

Note Hilbert's 5th problem deals with the question of to what extent a topological group has a differential structure. This problem has many interpretations. One of the most important of them was solved by Gleason, Montgomery-Zippin and Yamabe among other contributions: *every connected locally compact topological group without small subgroups (a neighborhood of the identity does not contain a subgroup other than the trivial subgroup) is a Lie group.*

II.1.1 The Maurer-Cartan form

Given a Lie group G and its Lie algebra $\mathfrak{g}$, one might wonder how $\mathfrak{g}$ controls the full tangent space $\mathrm{T}G$. Since G is a group, we can always translate T_eG to any T_gG by doing a left translation L_g or a right translation R_g. We choose to identify any tangent space T_gG with the left translation $(L_g)_*\mathrm{T}_eG$. It implies that $\mathrm{T}G$ has a parallelism $\mathrm{T}G \to G \times \mathfrak{g}$. This parallelism is determined by the Maurer-Cartan form.

Definition II.9 *A manifold M^n is parallelisable if there exists n vector fields $(X_1, \ldots, X_n)$ such that at each point $p \in M$, $(X_1(p), \ldots, X_n(p))$ is a basis of T_pM.*

Definition II.10 *The (left) Maurer-Cartan form on a Lie group G is the $\mathfrak{g}$-valued 1-form θ defined by*

$$\forall X_g \in \mathrm{T}_gG,\ \theta(X_g) = (L_g)_*^{-1}(X_g) \in \mathfrak{g}. \tag{17}$$

Note Let X be a vector field on G, then $\theta(X) = v$ is constant, if and only if, X is left-invariant and $X(g) = (L_g)_* v$. It furnishes a parallelism of G by choosing a basis of $\mathfrak{g}$.

Recall that for any 1-form α we have

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]). \tag{18}$$

Proposition II.11 (Structural equation) *For any $X, Y \in \mathrm{T}_gG$,*

$$d\theta(X,Y) + [\theta(X), \theta(Y)] = 0. \tag{19}$$

Proof We can evaluate $d\theta(X,Y)$ by assuming that X, Y are prolongated by left-invariant vector fields X^* and Y^*. For any left-invariant vector field X^*, the image by the Maurer-Cartan form is constant on it by definition. Therefore $X^*(\theta(Y^*))$ and $Y^*(\theta(X^*))$ are both zero. Moreover, since X^*, Y^* are left-invariant, so is $[X^*, Y^*]$ and therefore $\theta([X^*, Y^*]) = [\theta(X), \theta(Y)]$. ▧

Maurer-Cartan form with coordinates The choice of a basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$ allows to write $\theta = (\theta^1, \ldots, \theta^n)$ by duality. With X_i the left-invariant vector field verifying $\theta(X_i) = e_i$, we can determine the *structure coefficients*:

$$[X_i, X_j] = \sum_k c_{ij}^k X_k. \tag{20}$$

The structural equation becomes:

$$\mathrm{d}\theta^k(X,Y) = -\sum_{i<j} c^k_{ij}\theta^i \wedge \theta^j. \tag{21}$$

Note Here we use a convention which might be different in some cases (see [KN63] pg. 28) and is sometimes the cause of a factor of $\frac{1}{2}$ in the formula. In fact we define

$$\theta^1 \wedge \theta^2(X,Y) = \theta^1(X)\otimes\theta^2(Y) - \theta^1(Y)\otimes\theta^2(X) \tag{22}$$

in contrast with

$$\theta^1 \wedge \theta^2(X,Y) = \frac{1}{2}\left(\theta^1(X)\otimes\theta^2(Y) - \theta^1(Y)\otimes\theta^2(X)\right). \tag{23}$$

Example Consider the group $\mathrm{SO}(2) \subset \mathrm{GL}(2,\mathbf{R})$. This group is generated by:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{24}$$

By differentiating this parametrization, we obtain

$$\mathrm{d}g_\phi = \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix}\mathrm{d}\phi \tag{25}$$

Hence, the Lie algebra is also one dimensional and is generated by a single element:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{26}$$

The Maurer-Cartan form translates $\mathrm{d}g_\phi$ for any ϕ to $\mathrm{d}g_0$ by a left translation. Therefore it is given by

$$\theta_\phi = g(\phi)^{-1}\,\mathrm{d}g_\phi \tag{27}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix}\mathrm{d}\phi \tag{28}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathrm{d}\phi. \tag{29}$$

Matrix groups If $G \subset \mathrm{GL}(n,\mathbf{R})$ is a matrix group with Lie algebra $\mathfrak{g} \subset M_{n\times n}$ one can write the Maurer-Cartan form at $g \in G$ and it is given by $\theta_g = g^{-1}\,\mathrm{d}g$.

Here we interpret $\mathrm{d}g$ as the differential of the embedding of G into the space of matrices $M_{n\times n}$. In coordinates g_{ij} of that embedding, one has $\theta_g = g^{-1}_{ik}\,\mathrm{d}g_{kj}$, which is a $\mathfrak{g}$-valued 1-form.

Vector space valued forms The Maurer-Cartan form is an example of vector space valued form. We define the wedge product of a V_1-valued 1-form θ_1 and a V_2-valued 1-form θ_2 to be the $V_1 \otimes V_2$-valued form

$$\theta_1 \wedge \theta_2(X,Y) = \theta_1(X)\otimes\theta_2(Y) - \theta_1(Y)\otimes\theta_2(X). \tag{30}$$

If there exists a bilinear map $[\cdot,\cdot] : V \times V \to V$ we note the composition of $\wedge$ (for 1-forms) and $[\cdot,\cdot]$ by

$$[\theta_1 \wedge \theta_2](X,Y) := [\theta_1(X),\theta_2(Y)] - [\theta_1(Y),\theta_2(X)]. \tag{31}$$

Observe then that $[\theta_1(X),\theta_2(Y)] = \frac{1}{2}[\theta_1 \wedge \theta_2](X,Y)$.

Writing, in general, θ_n for a $\mathfrak{g}$-valued n-form we may define the exterior derivative and the product of two forms accordingly. We easily verify:

(1) $[\theta_p \wedge \theta_q] = (-1)^{pq}[\theta_q \wedge \theta_p]$,

(2) $(-1)^{pr}[[\theta_p \wedge \theta_q] \wedge \theta_r] + (-1)^{qr}[[\theta_r \wedge \theta_p] \wedge \theta_q] + (-1)^{qp}[[\theta_q \wedge \theta_r] \wedge \theta_p]$.

Moreover,

$$\mathrm{d}[\theta_p \wedge \theta_q] = [\mathrm{d}\theta_p \wedge \theta_q] + (-1)^{pq+1}[\theta_p \wedge \mathrm{d}\theta_q]. \tag{32}$$

Darboux derivatives

A Maurer-Cartan form allows the computation of Darboux derivatives.

DEFINITION II.12 *If $f\colon M \to G$ is smooth and if θ is the Maurer-Cartan form of G then the Darboux derivative of f is:*

$$f^*\theta = \theta f_*. \tag{33}$$

Example 1 In $\mathbf{R}^n$ the Darboux derivative is in a sense closer to the usual derivative than the differential. Indeed, recall that if $f\colon \mathbf{R}^p \to \mathbf{R}^n$ is smooth, then

$$\forall (x, v) \in \mathrm{T}\mathbf{R}^n,\ f_*(x, v) = (f(x), \mathrm{d}f_x(v)). \tag{34}$$

The consideration of f_* or even $\mathrm{d}f$ depends strongly on the consideration of a base point. But with the Darboux derivative, the tangent spaces are connected:

$$f^*\theta(x, v) = \theta(f(x), \mathrm{d}f_x(v)) = [\mathrm{d}f_x(v)] \tag{35}$$

and the class of $[\mathrm{d}f_x(v)]$ belongs to a single copy of $\mathbf{R}^n$.

Example 2 One parameter subgroups of a group G are defined by elements of the Lie algebra. For any $x \in \mathfrak{g}$ one defines a homomorphism

$$\exp_x\colon \mathbf{R} \to G, \tag{36}$$

which is the unique homomorphism satisfying $\exp_x^*\theta = x$.

DEFINITION II.13 *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is defined by*

$$\exp(x) = \exp_x(1). \tag{37}$$

Although exp has several properties analogous to the real exponential, due to the non-commutativity, one has a more complicated formula for the product of two exponentials (it is the Baker-Campbell-Hausdorff formula which is only valid locally):

$$\exp(x)\exp(y) = \exp\left(x + y + \frac{1}{2}[x, y] + \cdots\right). \tag{38}$$

If $\phi\colon H \to G$ is a group homomorphism one has

$$\exp \circ \mathrm{d}\phi_e = \phi \circ \exp_e. \tag{39}$$

LEMMA II.14 *Let X be a left-invariant vector field. Then its flow is $R_{\exp(tx)}$ with $x = \theta(X)$.*

PROOF Since X is left-invariant, so must be its flow. Therefore the integral curve at $g \in G$ is given by $L_g \exp(tx) = R_{\exp(tx)} g$. Hence the flow is given by $R_{\exp(tx)}$. ∎

II.1.2 The adjoint representation

An action of a Lie group G on a manifold induces a representation of the group on the automorphism group of the tangent space of a fixed point of the action. For, let $\phi\colon G \times M \to M$ be an action with a fixed point $G \cdot p = p$ at $p \in M$. Then for every $g \in G$, $\mathrm{d}\phi_{(g,p)}$ acts on $\mathrm{T}_p M$ as a linear isomorphism. It furnishes a representation $g \mapsto \mathrm{d}\phi_{(g,p)} \in \mathrm{Aut}(\mathrm{T}_p M)$.

In particular the adjoint action $G \times G \to G$ defined by $(g, h) \mapsto ghg^{-1}$ induces the representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ (observe that $\mathrm{Aut}(\mathrm{T}_e G)$ is isomorphic to $\mathrm{GL}(n, \mathbf{R})$ with $n = \dim_{\mathbf{R}} G$). For $g \in G$, Ad_g is the automorphism

$$\mathrm{Ad}_g(X) = \mathrm{d}(h \mapsto ghg^{-1})_e(X) = (L_g)_*(R_{g^{-1}})_* X \tag{40}$$

The adjoint representation is also exactly what we need to compare the Maurer-Cartan form θ defined by left-invariance with the action by right translations.

Proposition II.15 *For any $g \in G$, the Maurer-Cartan form θ verifies*

$$R_g^*\theta(X) = \mathrm{Ad}_g^{-1}(\theta(X)). \tag{41}$$

Proof Assume that $X = (L_x)_* v$. By the preceding definition, we have:

$$\begin{aligned} R_g^*\theta(X) &= \theta((R_g)_* X) && (42)\\ &= \theta((R_g)_*(L_x)_* v) && (43)\\ &= \theta((L_x)_*(R_g)_* v) && (44)\\ &= \theta((R_g)_* v) && (45)\\ &= (L_g)_*^{-1}(R_g)_* v = \mathrm{Ad}_g^{-1} v. && (46)\end{aligned}$$

▧

The differential of Ad_g at the origin $g = e$ is denoted by $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathrm{T}_e G)$:

$$\mathrm{ad}_x = \mathrm{dAd}_e(x). \tag{47}$$

It is in fact given by the bracket of the Lie algebra.

Lemma II.16 *Let $x, y \in \mathfrak{g} \cong \mathrm{T}_e G$. Then*

$$\mathrm{dAd}_e(x, y) = \mathrm{ad}_x(y) = [x, y]. \tag{48}$$

Proof Let X, Y be the left-invariant vector fields prolongating x and y. That is to say, $\theta(X) = x$ and $\theta(Y) = y$ with θ the Maurer-Cartan form. First, observe the identity:

$$\mathrm{d}^2\mathrm{id}(V, W) = V(\mathrm{id}(W)) - W(\mathrm{id}(V)) - [V, W] \tag{49}$$

showing that, since $\mathrm{d}^2\mathrm{id} = 0$,

$$[V, W] = V(\mathrm{id}(W)) - W(\mathrm{id}(V)). \tag{50}$$

On the other hand we have

$$\mathrm{dAd}(X, Y) = X(\mathrm{Ad}(Y)) - Y(\mathrm{Ad}(X)) - \mathrm{Ad}([X, Y]) \tag{51}$$

We prove that

$$X(\mathrm{Ad}(Y))|_e = 2X(\mathrm{id}(Y))|_e. \tag{52}$$

Recall that the flow ϕ^t of X is $R_{\exp(tX)}$. Hence on one hand:

$$\begin{aligned} X(\mathrm{id}(Y))|_e &= \lim_{t=0} \frac{(\phi_e^{-t})_* Y(\phi_e^t(e)) - Y(e)}{t} && (53)\\ &= \lim_{t=0} \frac{(R_{\exp(-tX)})_*(L_{\exp(tX)})_* Y(e) - Y(e)}{t} && (54)\\ &= \lim_{t=0} \frac{\mathrm{Ad}(\exp(tX))_* Y(e) - Y(e)}{t} && (55)\end{aligned}$$

and on the other hand (note that $\mathrm{Ad}_\phi(Y)$ is again left-invariant):

$$\begin{aligned} X(\mathrm{Ad}(Y))|_e &= \lim_{t=0} \frac{(\phi_e^{-t})_* \mathrm{Ad}_{\phi_e^t(e)}(Y)(\phi_e^t(e)) - Y(e)}{t} && (56)\\ &= \lim_{t=0} \frac{(R_{\exp(-tX)})_*(L_{\exp(tX)})_*(L_{\exp(tX)})_*(R_{\exp(-tX)})_* Y(e) - Y(e)}{t} && (57)\\ &= \lim_{t=0} \frac{\mathrm{Ad}(\exp(2tX))_* Y(e) - Y(e)}{t} && (58)\\ &= 2X(\mathrm{id}(Y))|_e. && (59)\end{aligned}$$

To conclude, we observe that at $e \in G$:

$$\begin{aligned} \mathrm{dAd}(X, Y)|_e &= X(\mathrm{Ad}(Y)) - Y(\mathrm{Ad}(X)) - \mathrm{Ad}([X, Y]) && (60)\\ &= 2X(\mathrm{id}(Y))|_e - 2Y(\mathrm{id}(X))|_e - \mathrm{Ad}_e[X, Y]|_e && (61)\\ &= 2X(\mathrm{id}(Y))|_e - 2Y(\mathrm{id}(X))|_e - [X, Y]|_e && (62)\\ &= [X, Y]|_e. && (63)\end{aligned}$$

▧

More generally, we have:

Proposition II.17 *The differential of the representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{T}_e G)$ *at* $g \in G$ *computed at the vector* $X^* = (L_g)_* X \in \mathrm{T}_g G$ *is*

$$\mathrm{dAd}_g(X)(Y) = \mathrm{Ad}_g(\mathrm{ad}_X(Y)). \tag{64}$$

Proof Writing a path through g as $L_g\gamma(t)$ with $\gamma(0) = e$ and $\dot\gamma(0) = X$ we have $\mathrm{Ad}_{L_g\gamma(t)}(Y) = \mathrm{Ad}_g \circ \mathrm{Ad}_{\gamma(t)}(Y)$. Therefore

$$(\mathrm{dAd}_g(X))(Y) = \left.\frac{\mathrm{dAd}_g \circ \mathrm{Ad}_{\gamma(t)}}{\mathrm{d}t}\right|_{t=0}(Y) = \mathrm{Ad}_g \circ \mathrm{ad}_X(Y). \tag{65}$$

▧

The adjoint automorphism by $g \in G$ fits in the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ \big\downarrow \exp & & \big\downarrow \exp \\ G & \xrightarrow{g(\cdot)g} & G \end{array} \tag{66}$$

and the adjoint representation satisfies

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ \big\downarrow \exp & & \big\downarrow \exp \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}) \end{array} \tag{67}$$

II.2 Homogeneous spaces

Homogeneous spaces will be the flat model geometries. They appear naturally when there exists a transitive action. Indeed, if $G \times M \to M$ is a transitive action one can identify M with the quotient ${}^{G}\!/_{H_x}$ where H_x is the isotropy subgroup of a chosen element $x \in M$. A different choice $gx \in M$ gives rise to the isotropy $H_{gx} = gH_x g^{-1}$.

Definition II.18 *A homogeneous space is a differential manifold obtained by the quotient of a Lie group* G *by a closed Lie subgroup* $H \subset G$*. We note the set of left cosets* gH *by* ${}^{G}\!/_{H}$*.*

The group G acts transitively on the homogeneous space ${}^{G}\!/_{H}$ by left translations, the isotropy subgroup at the identity being H.

Note If H were not closed then the quotient ${}^{G}\!/_{H}$ would not separated with the quotient topology.

Examples

(1) *The Euclidean space.*

The group of the isometries of the Euclidean space is $\mathrm{Eucl} = \mathbf{R}^n \rtimes \mathrm{O}(n)$. It acts on $\mathbf{R}^n$ with isotropy $\mathrm{O}(n)$. Therefore $\mathbf{R}^n = {}^{\mathrm{Eucl}}\!/_{\mathrm{O}(n)}$ as homogeneous space.

(2) *The hyperbolic space.*

Hyperbolic space is the simply connected complete constant negative sectional curvature Riemannian space. Its connected isometry group is $\mathrm{SO}(n,1)$ with isotropy $\mathrm{SO}(n)$. Here $\mathrm{SO}(n,1)$ is the group preserving the quadratic form

$$\begin{pmatrix} \mathrm{id}_{\mathbf{R}^n} & 0 \\ 0 & -1 \end{pmatrix}. \tag{68}$$

(3) *The similarity group acting on* $\mathbf{R}^n$.

The connected similarity group is the group $\mathrm{Sim}(\mathbf{R}^n) = \mathbf{R}^n \rtimes (\mathbf{R}_+^* \times \mathrm{O}(n))$. It is a subgroup of the affine group $\mathrm{Aff}(\mathbf{R}^n)$. Transformations of $\mathbf{R}_+^* \times \mathrm{O}(n)$ are of the form $\lambda P(x)$ with $\lambda > 0$ and P an orthogonal transformation.

The similarity group is the conformal group acting on $\mathbf{R}^n$. (Each conformal transformation has to be defined on the full space $\mathbf{R}^n$.) Therefore, it consists of the transformations of $\mathbf{R}^n$ which preserve angles. The isotropy at the origin is $\mathbf{R}_+^* \times \mathrm{O}(n)$.

(4) *The conformal sphere.*

There are more conformal transformations than just $\mathrm{Sim}(\mathbf{R}^n)$. But those are not defined strictly on $\mathbf{R}^n$ but rather on the one-point compactification S^n. The conformal sphere is the homogeneous space $\mathrm{PO}(n+1,1)/\mathrm{Sim}(\mathbf{R}^n)$.

(5) *The projective space.*

The projective space $\mathbf{RP}^n$ is the homogenous space $\mathrm{GL}(n+1,\mathbf{R})/H$ where

$$H = \left\{ \begin{pmatrix} \star & \star \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{GL}(n,\mathbf{R}) \right\}. \tag{69}$$

(6) *Flag spaces.*

The projective space is an example of flag spaces. A flag is a sequence $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbf{F}^n$ for any field $\mathbf{F}$. For instance, the projective space $\mathbf{FP}^n$ is the set of lines in $\mathbf{F}^{n+1}$.

A complete flag is a flag with $\dim V_i = i$. They are maximal in length. When $\mathbf{F} = \mathbf{C}$ we get an homogeneous space structure with the quotient

$$\mathrm{SU}(n) \big/ \mathrm{S}(\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)). \tag{70}$$

(7) *Stiefel manifolds.*

The space of orthonormal k-frames in $\mathbf{R}^n$ (with $0 < k < n$) is the Stiefel manifold $S(k,n)$. It is possible to show that

$$S(k,n) = \mathrm{SO}(n) \big/ \mathrm{SO}(n-k). \tag{71}$$

(8) *Every manifold is a homogeneous space.*

The full group of the diffeomorphisms of a manifold is not a Lie group but might be described by an analogous structure with infinite dimension.

The easiest situation is for a compact manifold, say M. The smooth diffeomorphism group $\mathrm{Diff}^\infty(M)$ has a structure of a Fréchet Lie group which is homeomorphic to the space of smooth vector fields. The group $\mathrm{Diff}^\infty(M)$ acts transitively on M. Therefore, any manifold can be considered as a homogeneous space $\mathrm{Diff}^\infty(M)/H$, where H is the isotropy at a point in M, that is to say, the set of diffeomorphisms fixing the point. We will not deal with infinite dimension Lie groups.

Construction à la Cartan We can reproduce how Cartan described the construction of the Maurer-Cartan form at the early stages of the theory. In fact, we here describe the main technique of the moving frame (*repère mobile*) that Cartan attributes to Darboux.

Consider the affine space $\mathbf{R}^3$. At any point $m \in \mathbf{R}^3$, associate a frame (e_1, e_2, e_3) base at m. The map (e_1, e_2, e_3) should be smooth depending on m.

The infinitesimal change of m by δm can be expressed by:

$$\delta m = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3. \tag{72}$$

It gives a 1-form with values in $\mathbf{R}^3$.

The infinitesimal change of a base vector e_i by δe_i can be described by the image of an infinitesimal matrix acting on (e_1, e_2, e_3):

$$\delta e_i = \omega_i^1 e_1 + \omega_i^2 e_2 + \omega_i^3 e_3 \tag{73}$$

and this furnishes a 1-form with values in $\mathfrak{gl}(3)$.

Those four 1-forms $\theta = (\delta e_1, \delta e_2, \delta e_3, \delta m)$ compose the Maurer-Cartan form of the affine space.

II.2.1 The tangent space

With a homogeneous space ${}^{G}\!/_{H}$ the tangent space can be described infinitesimally and the action of G (on the left) can be measured.

At eH, the tangent space is naturally isomorphic to ${}^{\mathfrak{g}}\!/_{\mathfrak{h}}$ as linear spaces. Therefore, the tangent bundle of the homogenous spaces $\mathrm{T}{}^{G}\!/_{H}$ can be seen as a quotient of the trivial bundle

$$G \times_H {}^{\mathfrak{g}}\!/_{\mathfrak{h}}. \tag{74}$$

The quotient will be by the right action of H:

$$(g, v) \cdot h \sim (gh, \mathrm{Ad}(h)^{-1} v). \tag{75}$$

Note that at the isotropy $H \subset G$, the action of $h \in H$ on a point pH is $hpH = hph^{-1}H$ and therefore H acts on $\mathrm{T}_{eH}{}^{G}\!/_{H}$ by $\mathrm{Ad}(h)$.

Proposition II.19 *There exists a canonical isomorphism*

$$\mathrm{T}{}^{G}\!/_{H} \cong G \times_H {}^{\mathfrak{g}}\!/_{\mathfrak{h}}. \tag{76}$$

Proof Let $\pi\colon G \to {}^{G}\!/_{H}$ be the quotient map. Let $\phi\colon G \times {}^{\mathfrak{g}}\!/_{\mathfrak{h}} \to \mathrm{T}{}^{G}\!/_{H}$ be defined by

$$\phi(g, v) = (gH, \pi_*(L_g)_* v). \tag{77}$$

We prove that this map is well defined in the quotient by the right action of H. Note that $\pi_*(R_h)_* = \pi$ since $\pi \circ R_h = \pi$ and $\pi_*(L_g)_* = (L_g)_* \pi_*$.

$$\phi((g, v) \cdot h) = \phi(gh, \mathrm{Ad}(h)^{-1} v) \tag{78}$$

$$= (ghH, \pi_*(L_{gh})_* \mathrm{Ad}(h)^{-1} v) \tag{79}$$

$$= (gH, (L_g)_* \pi_*(R_h)_* v) \tag{80}$$

$$= (gH, (L_g)_* \pi_* v) = \phi(g, v) \tag{81}$$

We can check that this morphism is injective at every point. If $\phi(g, v) = (gH, 0)$ then $\pi_* v = 0$ and therefore $v \in \mathfrak{h}$. It is surjective by dimensionality. ▩

II.2.2 Effective pairs

It is important to keep track of both groups G and H and not only their quotient space. On the other hand it is reasonable to consider only connected quotients ${}^{G}\!/_{H}$.

Definition II.20 *We will refer as a Klein geometry a pair* (G, H) *such that the homogeneous space* ${}^{G}\!/_{H}$ *is connected.*

There are two conditions which one can add without much loss of generality, namely, that the action of G be effective and that G be connected.

Note that $g \in G$ acts trivially on ${}^{G}\!/_{H}$ then $geH = eH$ and therefore $g \in H$. Denote $h \in H$ acting trivially. For any $g \in G$ and any coset pH we would have that $ghg^{-1}pH = g(h(g^{-1}pH))$ is equal to $g(g^{-1}pH)$ since h acts trivially on $g^{-1}pH$ and therefore $ghg^{-1}pH = pH$. So if h acts trivially, so does ghg^{-1}.

Definition II.21 *We say that a maximal subgroup $K \subset H$ which is normal in G is the kernel of a Klein geometry. The action of K is trivial and we say that the geometry is effective if $K = \{e\}$.*

If K is the maximal normal subgroup in H (the definition implies that K is a closed subgroup of G) one can consider the effective geometry $({}^{G}\!/_{K}, {}^{H}\!/_{K})$ which describes the same homogeneous space as ${}^{({}^{G}\!/_{K})}\!/_{({}^{H}\!/_{K})}$. It is diffeomorphic to ${}^{G}\!/_{H}$ with an equivariant action by ${}^{G}\!/_{K}$.

Sometimes one might consider non-effective Klein geometries. For instance, ${}^{SL(2,\mathbf{R})}\!/_{SO(2)}$ corresponds to the hyperbolic geometry but the subgroup $\mathbf{Z}_2 \subset SL(2,\mathbf{R})$ generated by $-\mathrm{id}$ is a maximal normal subgroup contained in $SO(2)$. Nonetheless, this subgroup is discrete and is does not intervene infinitesimally.

If G is not connected one can consider the connected component containing the identity $G_e \subset G$ and we obtain that ${}^{G}\!/_{H}$ is diffeomorphic to ${}^{G_e}\!/_{(H\cap G_e)}$ with an equivariant action by G_e. This follows since if ${}^{G}\!/_{H}$ is connected, one has $G = G_e H$. On the other hand, one can prove that if H is connected then G is also connected.

Lemma II.22 *Let $N \subset G$ be a normal subgroup with corresponding algebras $\mathfrak{n} \subset \mathfrak{g}$. Then for all $v \in \mathfrak{g}$ and $n \in N$,*

$$\mathrm{Ad}_n(v) - v \in \mathfrak{n}. \tag{82}$$

Proof Since N is normal, for any $g \in G$ and any $n \in N$ we have $ngn^{-1}g^{-1} \in N$. Let $g(t) = \exp(tv)$. We have:

$$(L_n L_{g(t)} R_{n^{-1}})g(-t) \in N \tag{83}$$

and by derivation at $t = 0$:

$$\mathrm{Ad}_n(v) - v \in \mathfrak{n}. \tag{84}$$

▩

Reciprocally, this condition implies, by differentiation along a path in N, that $[\mathfrak{n},\mathfrak{g}] \subset \mathfrak{n}$ so $\mathfrak{n}$ is an ideal of G.

We will need to identify maximal normal subgroups of G contained in $H \subset G$. The goal is to obtain properties for effective Klein geometries. The easiest way to start is with a normal subgroup N of H ($N = H$ is the most natural choice) so that its Lie algebra $\mathfrak{n}$ is an ideal of $\mathfrak{h}$. According to the preceding lemma, a candidate for a normal subgroup of G contained in $N \subset H$ is

$$N' = \{n \in N \mid \forall v \in \mathfrak{g},\ \mathrm{Ad}_n v - v \in \mathfrak{n}\}. \tag{85}$$

The subgroup N' might be much smaller that N. At least, it is still normal in H:

$$\mathrm{Ad}_{hnh^{-1}}(v) - v = \mathrm{Ad}_h(\mathrm{Ad}_n \mathrm{Ad}_{h^{-1}}(v) - \mathrm{Ad}_{h^{-1}}(v)) \in \mathrm{Ad}_h(\mathfrak{n}) \subset \mathfrak{n}. \tag{86}$$

The greatest normal subgroup of G which is contained in H is obtained by the following procedure.

Proposition II.23 *Suppose G is connected and $H \subset G$ a closed Lie subgroup. Define the decreasing sequence of subgroups of H:*

$$N_0 = H, \tag{87}$$

$$\forall i \geq 0,\ N_{i+1} = \{n \in H \mid \mathrm{Ad}_n v - v \in \mathfrak{n}_i,\ \forall v \in \mathfrak{g}\}. \tag{88}$$

Then, each $N_i \subset H$ is a closed normal subgroup of H and the intersection

$$N_\infty = \bigcap_i N_i \subset H \tag{89}$$

is the largest normal subgroup of G contained in H.

PROOF The fact that N_i and N_∞ are normal will depend on the following computation, related to the preceding paragraph. Let $n \in G$, $g \in G$ and $k \geq 0$. Assume that $\mathrm{Ad}_n\, v = v + w(v)$ for any $v \in \mathfrak{g}$, with a corresponding $w(v) \in \mathfrak{n}_k$. Then

$$\mathrm{Ad}_{gng^{-1}}\, v = \mathrm{Ad}_g \mathrm{Ad}_n(\mathrm{Ad}_{g^{-1}}\, v) \tag{90}$$

$$= \mathrm{Ad}_g \left(\mathrm{Ad}_{g^{-1}}\, v + w(\mathrm{Ad}_{g^{-1}}(v))\right) \tag{91}$$

$$= v + \mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))). \tag{92}$$

Now, to see that each group N_i is normal in H, note that if $n \in N_i$ and $g \in H$ then the preceding computation shows that gng^{-1} belongs to N_i if, and only if, $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_{i-1}$. By hypothesis, $w(\mathrm{Ad}_{g^{-1}}(v)) \in \mathfrak{n}_{i-1}$. By recurrence, $\mathrm{Ad}_g(\mathfrak{n}_{i-1}) \subset \mathfrak{n}_{i-1}$, showing that we have indeed $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_{i-1}$.

It is clear that N_∞ is well defined and is normal in H. We have to show it is also normal in G. First, $\mathfrak{n}_\infty \subset \mathfrak{g}$ is an ideal. Indeed, by differentiation of $\mathrm{Ad}_n(v) = v + w(v)$ along a path $n(t)$ we have $[n, v] = w'(v)$ and it belongs to to $\mathfrak{n}_\infty$ since $w(v)$ does.

Since $\mathfrak{n}_\infty \subset \mathfrak{g}$ is an ideal and G is connected, the component of the identity of N_∞ is normal in G. But then it implies $\mathrm{Ad}_{g^{-1}}\, \mathfrak{n}_\infty = \mathfrak{n}_\infty$. By the preceding computation it implies $\mathrm{Ad}_g(w(\mathrm{Ad}_{g^{-1}}(v))) \in \mathfrak{n}_\infty$ and therefore that N_∞ is indeed normal.

To complete the proof, we show that for a normal subgroup $N \subset G$ contained in H, $N \subset N_\infty$: by induction, $N \subset H$ and if $N \subset N_i$ so $\mathfrak{n} \subset \mathfrak{n}_i$ and therefore $N \subset \left\{n \in H \mid \mathrm{Ad}_n\, v - v \in \mathfrak{n}_i,\ \forall v \in \mathfrak{g}\right\} = N_{i+1}$. ▨

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