REDUNDANCY OF HYPERBOLIC TRIANGLE GROUPS IN SPHERICAL CR REPRESENTATIONS

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Abstract

Falbel, Koseleff and Rouillier computed a large number of boundary unipotent CR representations. Those representations are not always discrete. By experimentally computing their limit set, one can determine that those with fractal limit sets are discrete. Most of those discrete representations can be classified into $(3,3,n)$ complex hyperbolic triangle groups. By exact computations, we verify the existence of those triangle representations, which have unipotent boundary holonomy. We also show that many representations are redundant: for n fixed, all the $(3,3,n)$ representations encountered are conjugated and only one among them is uniformizable.

CONTENTS

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I — Introduction

We are interested in triangle groups

$$\Lambda(p,q,r) = \left\langle a,b,c\,; \quad \begin{matrix} a^2 = b^2 = c^2 = e, \\ (ab)^p = (bc)^q = (ca)^r = e \end{matrix} \right\rangle \tag{1}$$

and how they can be represented into the Lie group $\mathrm{PU}(2,1)$ by complex reflections, that is to say, with a, b and c all being complex reflections with respect to complex geodesic lines in the complex hyperbolic plane $\mathbf{H}^2_{\mathbf{C}}$. Such a representation is called a *complex hyperbolic triangle group*, denoted by $\Delta(p,q,r)$, and the images of a, b and c are often denoted by I_1, I_2 and I_3. An additional hypothesis is that $\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{r}<\pi$ and can be interpreted as the requirement for the triangle to lie in the hyperbolic plane.

Triangle groups represented in $\mathrm{PO}(2,1)$ (which is the transformation group of the *real* hyperbolic plane) are fully prescribed by p, q and r (up to conjugation), whereas in $\mathrm{PU}(2,1)$ (which is the transformation group of the *complex* hyperbolic plane) an additional parameter controls the representation. This additional parameter can be interpreted as follows. One can always set two vertices of the triangle in a same real plane of $\mathbf{H}^2_{\mathbf{C}}$. The last vertex has to be placed at the intersection of the complex geodesic lines issued from the previous vertices. That intersection is a one-dimensional topological space and it represents the possible values of the additional parameter. Only one of those values corresponds to the case where the last vertex lies in the same previous real plane (and therefore corresponds to a $\mathbf{R}$-Fuchsian representation). This parameter is called the *angular invariant.*

Complex hyperbolic triangle groups are a very rich class of representations in $\mathrm{PU}(2,1)$ and one can ask whether it covers many known representations. In particular, a large number of fundamental groups' representations of knots' and links' complements are known in $\mathrm{PU}(2,1)$. Falbel, Koseleff and Rouillier [FKR15] explicitly computed those representations with unipotent boundary for complements described by four or less tetrahedra. The additional hypothesis of unipotent boundary is strong but allows this explicit computation that remains a highly complex numerical problem.

Those representations are accompanied by some delicate questions. *Which representations are discrete? Which are complex hyperbolic triangle groups?*

In the early stages of those researches, Falbel [Fal08] constructed the unipotent boundary representations of the figure-eight knot. Those (essentially) three representations have two discrete representations among them, which are indeed complex hyperbolic triangle groups. To be more specific, those two representations can be identified with the even-length words' normal subgroup of a complex hyperbolic triangle group $\Delta(3,3,4)$.

When complex hyperbolic triangle groups were exposed by Schwartz [Sch02] in an ICM talk, he proposed the following conjecture which allows to compare the previous two emphasized questions: if we find a complex hyperbolic triangle group, is it a discrete and injective representation?

Conjecture I.1 (Schwartz) *A complex hyperbolic triangle group $\Delta(p,q,r)$ is a discrete and injective representation, if and only if $I_2I_3I_1I_3$ and $I_1I_2I_3$ are both not elliptic.*

Note that, in some rare cases, $I_2I_3I_1I_3$ and $I_1I_2I_3$ can have finite order and $\Delta(p,q,r)$ can remain discrete (but is not an injective representation). For example, Thompson [Tho10] showed that there exists a representation $\Delta(3,3,4)$ with $I_2I_3I_1I_3$ of order

7 and a representation $\Delta(3,3,5)$ with $I_2I_3I_1I_3$ of order 5, and that both representations are lattices.

A first step toward this conjecture is a result of Grossi [Gro07]. It shows that in the case of $(p,q,r) = (3,3,n)$, the fact that $I_2I_3I_1I_3$ is not elliptic implies that $I_1I_2I_3$ is not either. A proof of Schwartz' conjecture in the case of $(3,3,n)$ has been given by Parker, Wang and Xie [PWX16] and the case of $(3,3,\infty)$ has been studied by Parker and Will [PW17].

Theorem I.2 ([PWX16],[PW17]) *Let $4 \leq n \leq \infty$. Let Γ be a hyperbolic $(3,3,n)$ triangle group. Then Γ is a discrete and faithful representation of $\Delta(3,3,n)$ if and only if $I_2I_3I_1I_3$ is not elliptic.*

This allows the following study: take ρ a boundary unipotent representation; is it discrete? Is it triangular? Both questions are here treated in a systematic and experimental manner.

To study the discreteness of a representation we chose to experimentally compute its limit set. This set is an attractor for the iteration dynamic and a simple argument allows to prove the discreteness: if this limit set is fractal then the representation is discrete (unfortunately, the converse does not hold).

Those representations with fractal limit sets are then set aside from the others. There are only two dozens of them (in comparaison of hundreds). They are then manipulated in order to try to prove that they come from a complex hyperbolic triangle group. We show that many of them are in fact triangle groups with $(p,q,r) = (3,3,n)$ with $n \geq 4$.

Since those representations are discrete, $I_2I_3I_1I_3$ is not elliptic. In our examples, we show that it is even parabolic unipotent and that the conjugacy class can be chosen so that $I_2I_3I_1I_3$ generates the boundary holonomy of the fundamental group.

We would like to stress this last phenomenon. When a manifold M has an abstract triangle representation $\rho\colon \pi_1(M) \to \Lambda(p,q,r)$, then this representation has various embeddings $\pi_1(M) \to \Lambda(p,q,r) \to \Delta(p,q,r) \subset \mathrm{PU}(2,1)$ by the choice of the angular invariant. Only one representation $\Delta(p,q,r)$ is such that $I_2I_3I_1I_3$ is parabolic unipotent. In every example that the author encountered, this choice implied that the boundary holonomy is also parabolic unipotent and even described by the element $I_2I_3I_1I_3$. If this phenomenon was always true, it would justify the search for triangle representations using only the computation of boundary unipotent CR representations. It is a drastic reduction. For example, the figure-eight knot has a two-dimensional character variety [Fal+16] but only three (up to complex conjugation) different unipotent boundary representations.

A phenomenon that will also interest us here is redundancy: among all the $(3,3,n)$ triangle groups' representations (with n fixed) appearing in the census in [FKR15], only one corresponds to a uniformization of the underlying CR structure.

There is a delicate relationship between a CR representation of a manifold M and a *uniformizable* CR representation of M. In the latter case, the image group Γ completely determines M: if U is its discontinuity domain then U/Γ is diffeomorphic to M (that is the definition of being uniformizable). Interestingly, it is very hard to determine whether a CR representation is uniformizable. The algebraic computations of the CR representations do not involve a conservation of the topological information.

Deraux [Der15] first showed that m009 and m015 are two manifolds with boundary unipotent CR representations $\Delta(3,3,5)$ which are conjugated, but only the representation of m009 is uniformizable. In fact, once we know that two representations are conjugated, then it follows that only one of them can be uniformizable by an evident topological argument.

The final result of the present work is:

Theorem (IV.1) *In the following table, the manifolds have a* $(3,3,n)$ *complex hyperbolic triangle group representation, with the normal subgroup of the even-length words for image. Furthermore, all those representations (for a shared a column) are the same, up to conjugation and complex conjugation. The starred manifolds correspond to the uniformizable representations.*

$\Delta(3,3,4)$	$\Delta(3,3,5)$	$\Delta(3,3,6)$	$\Delta(3,3,7)$	$\Delta(3,3,\infty)$
m004*	m009*	m023*	(m039)*	m129*
m022	m015		m032	m203
m029	m142		m045	
m034	m146			
m081				
m117				

The even-length subgroups of the triangle groups $\Delta(3,3,n)$ with $I_2 I_3 I_1 I_3$ parabolic unipotent were finely studied by a theorem of Acosta. This theorem allows to identify the manifold at infinity and to check which manifolds have a uniformizable triangle representation.

Theorem I.3 ([Aco19]) *Let* $4 \le n \le \infty$*. Let* Γ *be a hyperbolic* $(3,3,n)$ *triangle group. Suppose that* $I_2 I_3 I_1 I_3$ *is parabolic unipotent. Let* $\Gamma' \subset \Gamma$ *be the subgroup of even-length words. Then the manifold at infinity of* $\mathbf{H}^2_{\mathbf{C}}/\Gamma'$ *is the Dehn surgery with slope* $(1, n-3)$ *on any cusp of the Whitehead link complement.*

In section II, we succinctly expose a few elements of complex hyperbolic geometry that are necessary to the subject. In section III, we explain how the numerical experiments were driven. We also propose visual clues in order to recognize the limit sets of the various triangle representations $\Delta(3,3,n)$ with n varying and $I_2 I_3 I_1 I_3$ always parabolic unipotent. Finally, in section IV we summarized the various representations with an apparent fractal limit set. We show that a subclass is only composed of $\Delta(3,3,n)$ triangle groups. For this, we only employ formal computations, so the result is certain. We also explain how to retrieve which of each class for n fixed is uniformizable.

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II — Elements of complex hyperbolic geometry

In this first section, we expose the main tools and notions in use. One can compare with [Wil16], [Gol99], [Pra05] and [CG74].

We consider the space $\mathbf{C}^{2,1}$, that is to say $\mathbf{C}^3$ equipped with the Hermitian product

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3}. \tag{2}$$

The linear subspace of the vectors verifying $\langle z,z\rangle < 0$ can be projectivised in $\mathbf{CP}^2$ and is identified to the *complex hyperbolic plane*, $\mathbf{H}^2_{\mathbf{C}}$. In the affine chart $z_3 = 1$, one can identify $\mathbf{H}^2_{\mathbf{C}}$ with the set of vectors verifying $|z_1|^2 + |z_2|^2 < 1$. This description of the complex hyperbolic plane is also known as the Klein model of $\mathbf{H}^2_{\mathbf{C}}$. Its boundary in the complex projective plane is a differentiable sphere S^3 and is given by $|z_1|^2 + |z_2|^2 = 1$. Those points are in correspondance with the non-null vector lines in $\mathbf{C}^{2,1}$ verifying $\langle z,z\rangle = 0$.

The orthogonal group of $\mathbf{C}^{2,1}$ is $\mathrm{U}(2,1)$ and its projectivised version is $\mathrm{PU}(2,1)$. Together with the complex conjugation, the group $\widehat{\mathrm{PU}(2,1)}$ is the transformation group of $\mathbf{H}^2_{\mathbf{C}}$ and also of its boundary sphere. This last geometrical structure $(\widehat{\mathrm{PU}(2,1)}, S^3)$ is the *spherical CR structure* and $(\mathrm{PU}(2,1), S^3)$ is the *orientation-preserving spherical CR structure*.

Let $A \in \mathrm{SU}(2,1)$. If A has a fixed point in $\mathbf{H}^2_{\mathbf{C}}$ then A is *elliptic*. If $\inf\{d(x, A(x))\} > 0$ with d the associated distance function of $\mathbf{H}^2_{\mathbf{C}}$ then A is *loxodromic* (or *hyperbolic*). Otherwise, A is *parabolic*. One can determine the type of A by looking at its trace. We follow Goldman [Gol99] and let

$$f(\tau) = |\tau|^4 - 8\mathrm{Re}(\tau^3) + 18|\tau|^2 - 27. \tag{3}$$

If $f(\mathrm{tr}\,A) > 0$ then A is loxodromic, if $f(\mathrm{tr}\,A) < 0$ then A is elliptic (in fact regular elliptic: all its eigenvalues are different). When $f(\mathrm{tr}\,A) = 0$ there are three cases: if $\mathrm{tr}(A)^3 = 27$ then A is parabolic unipotent (all its eigenvalues are 1), otherwise it is either elliptic (and therefore a reflection with respect to a point or a complex geodesic) or ellipto-parabolic (a screw transformation along a complex geodesic). Note that when τ is real:

$$f(\tau) = (\tau+1)(\tau-3)^3, \tag{4}$$

and (under the hypothesis that $\mathrm{tr}(A)$ is real) A is therefore regular elliptic if $\mathrm{tr}(A) \in]-1,3[$, is loxodromic if $\mathrm{tr}(A) \notin [-1,3]$ and is parabolic unipotent if $\mathrm{tr}(A) = 3$.

Let M be a smooth manifold and $\pi_1(M)$ its fundamental group. A representation $\rho\colon \pi_1(M) \to \mathrm{PU}(2,1)$ is a (CR) *uniformization* of M if $U/\rho(\pi_1(M))$ is diffeomorphic to M, where $U \subset \partial\mathbf{H}^2_{\mathbf{C}}$ is the domain of discontinuity of $\rho(\pi_1(M))$. When ρ is discrete, $U = \partial\mathbf{H}^2_{\mathbf{C}} - L(\rho(\pi_1(M)))$, where $L(\rho(\pi_1(M)))$ is the *limit set* of $\rho(\pi_1(M))$. The next section will describe this set. A manifold admitting such a representation is said to be (CR) *uniformizable*. Those manifolds are of high matter in the study of spherical CR structures and are determined by the algebraic data of ρ. In general, $U/\rho(\pi_1(M))$ is a smooth manifold but is too hard to identify. It remains unknown which three-manifolds are CR uniformizable.

II.1 Limit sets

Let $\Gamma \subset \mathrm{PU}(2,1)$ be a subgroup. Its *limit set* $L(\Gamma)$ is given by:

$$L(\Gamma) = \overline{\Gamma\cdot p} \cap \partial\mathbf{H}^2_{\mathbf{C}}, \tag{5}$$

where $p \in \mathbf{H}^2_{\mathbf{C}}$ is any point ($L(\Gamma)$ is independent of this choice).

Lemma II.1 *The main properties of this set are the following. (Compare with [CG74].)*

(1) The limit set $L(\Gamma)$ is compact and Γ-invariant.

(2) If $A \subset \partial\mathbf{H}^2_{\mathbf{C}}$ is compact, Γ-invariant and is constituted of at least two points, then $A \subset L(\Gamma)$.
(3) If $L(\Gamma) = \emptyset$ then Γ fixes a point in $\mathbf{H}^2_{\mathbf{C}}$.
(4) If $L(\Gamma)$ has at most two points, then $L(\Gamma)$ is said to be elementary, *otherwise it has an infinity number of points and is perfect (each point is an accumulating point).*

An important result is the following.

Proposition II.2 ([CG74]) *If Γ is not discrete then $L(\Gamma)$ is either elementary, or equal to $\partial\mathbf{H}^2_{\mathbf{C}}$, or equal to the boundary of a totally geodesic proper subspace, that is to say a smooth circle.*

By consequence, if $L(\Gamma)$ is a fractal, then Γ is discrete. It is a powerful experimental way to check if Γ is discrete, since no abstract systematic argument allows to know this.

The auto-similarity property of limit sets can be justified by the following.

Lemma II.3 *Let Γ be a discrete subgroup of $L(\Gamma)$ and suppose that $L(\Gamma)$ is not elementary. Let $a \in L(\Gamma)$ be any point and V be any open neighbhorhood of a. Then there exists $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that*

$$L(\Gamma) = \bigcup \gamma_i \cdot V \cap L(\Gamma). \tag{6}$$

Proof Let $W = \partial\mathbf{H}^2_{\mathbf{C}} - \bigcup \Gamma \cdot V$. It is compact and Γ-invariant. By construction, W can not have more than one point. If $W = \{b\}$ then Γ let b fixed and it follows that $L(\Gamma)$ must be elementary since it is discrete (this relies on an observation on the corresponding Heisenberg transformation group). Therefore $W = \emptyset$ and it follows that $L(\Gamma) \subset \bigcup \Gamma \cdot V$. By compacity of $L(\Gamma)$, only a finite number of $\gamma_i \cdot V$ are necessary. ∎

II.2 Complex hyperbolic triangle groups

We will now describe more precisely the *complex hyperbolic triangle groups*

$$\Delta(p,q,r) = \left\langle I_1, I_2, I_3 \, ; \quad \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1 I_2)^p = (I_2 I_3)^q = (I_3 I_1)^r = e \end{array} \right\rangle \subset \mathrm{PU}(2,1), \tag{7}$$

with I_1, I_2 and I_3 all three being complex reflections. If p or q or r is infinite, then the corresponding relation vanishes.

Let $\Delta(p,q,r)$ be a non-singular complex hyperbolic triangle group (the geodesic lines corresponding to the reflections are distinct), with $2 \le p \le q \le r \le \infty$ and $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. Any such triangle group can be represented by a complex hyperbolic triangle in $\overline{\mathbf{H}^2_{\mathbf{C}}} \subset \mathbf{CP}^2$.

Let H_1, H_2 and H_3 be the vectorial hyperplanes of $\mathbf{C}^3$ covering the sides of the triangle in $\mathbf{CP}^2$. Let L_1, L_2 and L_3 be the dual complex lines of those hyperplanes defined by $\langle H_i, L_i \rangle = 0$. The group $\Delta(p,q,r)$ is fully described by them.

We only need to choose a base vector for each L_i in order to retrieve those lines. Furthermore, note that such base vectors form a basis of $\mathbf{C}^3$ since the triangle group is non-singular.

Let v_k be a base vector of L_k, then

$$I_k(x) = -x + \frac{2\langle v_k, x\rangle}{\langle v_k, v_k\rangle} v_k \tag{8}$$

is a complex reflection (note that $\langle v_k, x\rangle$ and not $\langle x, v_k\rangle$ makes this transformation linear) and verifies $I_k(h_k) = -h_k$ for any $h_k \in H_k$. That is to say, in $\mathbf{CP}^2$, $I_k(h_k) \equiv h_k$.

Therefore I_k indeed defines the reflection fixing H_k. Because $\langle v_k, v_k\rangle > 0$, one can normalize v_k so that $\langle v_k, v_k\rangle = 1$.

The last free parameters are an angle $z_k \in S^1$ for each v_k. One can set z_1 and then modify z_2 and z_3 so that $\langle v_1, v_2\rangle$ and $\langle v_2, v_3\rangle$ are real and positive. In general, $\langle v_1, v_3\rangle$ is not real and this lack can be measured by $\arg(\langle v_1, v_3\rangle)$. From an intrinsic point of view, that is to say without choosing the z_k's, the default for the vertices to be in a same real plane can be measured by

$$\theta = -\arg(\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle). \tag{9}$$

The value of θ is also known under the name of the *angular invariant*.

Once $\langle v_i, v_j\rangle = c_{ij}$ are known, it is easy to evaluate the matrices of I_1, I_2 and I_3 in the basis (v_1, v_2, v_3).

$$I_1 = \begin{pmatrix} 1 & 2c_{12} & 2c_{13} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{10}$$

$$I_2 = \begin{pmatrix} -1 & 0 & 0 \\ 2c_{21} & 1 & 2c_{23} \\ 0 & 0 & -1 \end{pmatrix} \tag{11}$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2c_{31} & 2c_{32} & 1 \end{pmatrix} \tag{12}$$

We still have to see how p, q, r and θ determine $\langle v_i, v_j\rangle = c_{ij}$. For the time being, we suppose $r < \infty$. In fact, the matrix given by the c_{ij}'s is equal to:

$$H = \begin{pmatrix} 1 & \cos\frac{\pi}{p} & \cos\frac{\pi}{r}e^{i\theta} \\ \cos\frac{\pi}{p} & 1 & \cos\frac{\pi}{q} \\ \cos\frac{\pi}{r}e^{-i\theta} & \cos\frac{\pi}{q} & 1 \end{pmatrix}. \tag{13}$$

And this shows that the c_{ij}'s fully determine p, q, r and θ in return. This matrix is an Hermitian form preserved by I_1, I_2 and I_3. The determinant of this matrix is given by

$$1 + 2\cos(\theta)\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r} - \cos\left(\frac{\pi}{p}\right)^2 - \cos\left(\frac{\pi}{q}\right)^2 - \cos\left(\frac{\pi}{r}\right)^2. \tag{14}$$

This determinant allows to decide when H has $(2,1)$ for signature. Since the trace of H is 3, it implies that at least one eigenvalue is positive. Therefore, its determinant is negative if and only if H has $(2,1)$ for signature. That is equivalent to:

$$\cos(\theta) < \frac{-1 + \cos\left(\frac{\pi}{p}\right)^2 + \cos\left(\frac{\pi}{q}\right)^2 + \cos\left(\frac{\pi}{r}\right)^2}{2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r}}. \tag{15}$$

That must be the case since the original Hermitian form $\langle\cdot,\cdot\rangle$ has $(2,1)$ for signature.

If $p = 2$ then c_{12} vanishes and one can make both c_{23} and c_{13} real. Therefore, $(2, q, r)$ complex hyperbolic triangle groups are rigid.

We now justify the expression of H. Up to conjugation, we can suppose that $H_1 \cap H_2$ is generated by $(0,0,1)$. This implies that v_1 and v_2 are both of the form $(x, y, 0)$. Therefore, every c_{ij} is given by $v_i^1\overline{v_j^1} + v_i^2\overline{v_j^2}$ since at least one of v_i or v_j has a vanishing third coordinate.

Therefore, geometrically speaking, c_{ij} is the cosine of the angle in $\mathbf{C}^2$ formed by the complex lines generated by the first two coordinates of v_i and v_j. It is the real part of c_{ij} that is equal to the cosine of the angle formed by the vectors given by the first two coordinates of v_i and v_j (see [Gol99, p. 36]).

Note that c_{13} is non real in general, but of course $\langle v_1, e^{i\theta} v_3\rangle = e^{-i\theta}\langle v_1, v_3\rangle = e^{-i\theta} c_{13}$ is real. The angle formed by H_1 and H_2 is equal to $\frac{\pi}{p}$ since $(I_1 I_2)^p = e$. By taking the duals v_1 and v_2, we get $c_{12} = \pm\cos\frac{\pi}{p}$ but we made c_{12} positive therefore $c_{12} = \cos\frac{\pi}{p}$. Likewise, $c_{23} = \cos\frac{\pi}{q}$ and $e^{-i\theta} c_{13} = \cos\frac{\pi}{r}$.

Finally, one can compute, with $i \neq j \neq k$:

$$\mathrm{tr}(I_i I_j I_k I_j) = 16|c_{ij} c_{kj}|^2 - 16\mathrm{Re}(c_{12} c_{23} c_{31}) + 4|c_{ik}|^2 - 1, \tag{16}$$

and note that in our conventions, $\mathrm{Re}(c_{12}c_{23}c_{31}) = c_{12}c_{23}c_{13}\cos\theta$. It shows that $\mathrm{tr}(I_i I_j I_k I_j)$ determines $\pm\theta$ once (p,q,r) is known. Since the complex conjugation changes θ into $-\theta$, we deduce from our discussion the following results. In the case where $(p,q,r) = (3,3,r)$ we have in fact:

$$\mathrm{tr}(I_i I_j I_k I_j) = 4\cos\left(\frac{\pi}{r}\right)^2 - 4\cos\frac{\pi}{r}\cos\theta. \tag{17}$$

Proposition II.4 ([Pra05]) *Let $3 \le p \le q \le r < \infty$ be such that $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. A representation of the triangle group*

$$\Delta(p,q,r) = \left\langle I_1, I_2, I_3\,;\quad \begin{matrix} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1I_2)^p = (I_2I_3)^q = (I_3I_1)^r = e \end{matrix}\right\rangle \tag{18}$$

into $\mathrm{PU}(2,1)$ *is determined by* $\theta = \arg(\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_1, v_3\rangle)$ *up to conjugation. Up to conjugation and complex conjugation, it is determined by* $\mathrm{tr}(I_i I_j I_k I_j)$*, with $i \neq j \neq k$.*

Furthermore, θ verifies

$$\cos(\theta) < \frac{-1 + \cos\left(\frac{\pi}{p}\right)^2 + \cos\left(\frac{\pi}{q}\right)^2 + \cos\left(\frac{\pi}{r}\right)^2}{2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r}} \tag{19}$$

and reciprocally, this condition suffices to define a representation with that value of θ.

The parameter θ can be taken in $]0,\pi]$ since the complex conjugation exchanges θ and $2\pi - \theta$. The possible values of $\mathrm{tr}(I_i I_j I_k I_j)$ are constrained by the preceding condition. For example, when $(p,q,r) = (3,3,r)$, we have

$$\cos\theta < \frac{-\frac{1}{2} + \cos\left(\frac{\pi}{r}\right)^2}{\frac{1}{2}\cos\frac{\pi}{r}} = \frac{-1 + 2\cos\left(\frac{\pi}{r}\right)^2}{\cos\frac{\pi}{r}}. \tag{20}$$

If we take a look at the trace of $I_i I_j I_k I_j$, its maximum is given for $\cos\theta$ minimal (that is to say -1) and its minimum by the maximum of $\cos\theta$.

$$\mathrm{tr}(I_i I_j I_k I_j) \le 4\cos\frac{\pi}{r}\left(\cos\frac{\pi}{r} + 1\right), \tag{21}$$

$$\mathrm{tr}(I_i I_j I_k I_j) > 4\cos\left(\frac{\pi}{r}\right)^2 - 4\cos\frac{\pi}{r}\left(\frac{-1 + 2\cos\left(\frac{\pi}{r}\right)^2}{\cos\frac{\pi}{r}}\right) = 4\left(1 - \cos\left(\frac{\pi}{r}\right)^2\right) > 0. \tag{22}$$

This computation shows that the range of values of $\mathrm{tr}(I_i I_j I_k I_j)$ is included in $\mathbf{R}_+$ and therefore, the range for which $\Delta(p,q,r)$ is discrete and injective is of the form $[3,m]$, with m the maximum stated before. The value 3 is indeed reachable: the minimum value for r is 4, since we have to verify $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$, and the maximum value of $4(1 - \cos(\frac{\pi}{r})^2)$ is indeed reached when r is minimal. This value for $r = 4$ is 1.

One can compute the angular invariant required to have $I_i I_j I_k I_j$ parabolic unipotent. It is given by:

$$\cos\theta = \cos\frac{\pi}{r} - \frac{3}{4\cos\frac{\pi}{r}}. \tag{23}$$

When one or several of p, q and r are non finite, we can get similar results by replacing the undefined c_{ij} with $\cosh(l_{ij}/2)$, where l_{ij} is the distance between the two complex hyperbolic geodesics H_i and H_j. See [Pra05]. In particular, it is still true that $\cos\theta$ is determined by $\mathrm{tr}(I_i I_j I_k I_j)$.

III — Experimental approach

In this section, we explain how we experimentally computed the limit sets of the representations appearing in the census of Falbel-Koseleff-Rouillier [FKR15]. We also propose a comparative experiment by simulating the limit sets associated to the $(3,3,n)$ triangle groups. It will allow us to propose visual clues in order to distinguish the different $(3,3,n)$ triangle groups' fractals when $I_2 I_3 I_1 I_3$ is parabolic.

The source code of the simulations and most of their results are available on the author's webpage. The code has been made open-source.

III.1 Computing limit sets

Lemma III.1 *Let $\Gamma \subset \mathrm{PU}(n,1)$ be a subgroup. Let Γ_L denote the subgroup of the loxodromic elements. Suppose that the limit set $L(\Gamma)$ is non elementary and that $\Gamma_L \neq \emptyset$. Then the closure of the accumulation points' set for the loxodromic iteration dynamic,*

$$\overline{\left\{x \mid \exists g \in \Gamma_L,\ \lim g^n x_0 = x\right\}},\ x_0 \in \mathbf{H}^2_{\mathbf{C}}, \tag{24}$$

is equal to the full limit set $L(\Gamma)$.

Proof This ensues from the following remark. Note that if g is loxodromic then any conjugation $\gamma g \gamma^{-1}$ is loxodromic too. Now, γx is equal to $\lim \gamma g^n \gamma^{-1}(x_0)$. This shows that the described set has two points (the attractive points of g and g^{-1}) and is Γ-invariant. ▧

It allows a first good strategy: computing attractive limit points of loxodromic elements. However, this strategy requires to compute a very large number of elements $g \in \Gamma$. This can be done by generating words of length n. If Γ is described by two generators, then there are approximately 3^n words of length n. An additional strategy consists in computing two lists of the half-length words and combining them at the last moment (in order to save a square root of memory space).

In practice, and this is particularly true with complex hyperbolic triangle groups, it is hard to get *different* points from such a computation. One can often see large concentrations of points in tiny boxes and even many copies of the same point. This is partly due to unknown relations between words, even at small length words.

Instead of only computing words and get their attractive limits, we used a second strategy in complement. When enough points are acquired, one can apply words on them (loxodromic or not) to get a better picture of the limit set. This method is much more efficient for it rarely makes redundant images. When nice symmetries are known (and for example, with complex hyperbolic triangles one knows the symmetries I_1, I_2 and I_3), this allows a much better result. To symmetrize fully, one can apply each symmetry successively on the set of points.

In practice, we first compute the attractive points of n_1-length words, then apply given symmetries on the set obtained, then apply n_2-length words on them, and again apply symmetries. At each step, we sort and select points in order to eliminate redundancy in the results.

At the end, we still have to project the points from $\mathbf{CP}^2$ into $\mathbf{R}^3 \subset \partial\mathbf{H}^2_{\mathbf{C}}$. It can be done once a Hermitian form determining $\partial\mathbf{H}^2_{\mathbf{C}}$ is known. We used a least-square method to solve the natural system of linear equations associated to such a Hermitian form in order to have this information.

III.2 Complex hyperbolic $(3,3,n)$ triangle groups

To compute the limit sets associated to PU(2,1)-representations of $(3,3,n)$ triangle groups, we used our previous parametrization of the reflections I_1, I_2, and I_3. We already have all the tools necessary to ensure that θ is acceptable and gives a discrete representation. Note that when $\theta = \pi$, it corresponds to a $\mathbf{R}$-Fuchsian representation and the limit set is therefore a ($\mathbf{R}$-)circle (since the representation preserves a real plane).

For $n \in \{4,5,6,7\}$ we show three projections of the limit set and an additional diagram proposing a visual clue to recognize the limit set (figures 1, 2, 3, and 4). This visual clue consists in looking for a pair of symmetric spikes and inspect the middle. We count the largest outer holes. When $n = 4$ there is none, when $n = 5$ there is one, when $n = 6$ there are two and when $n = 7$ there are three.

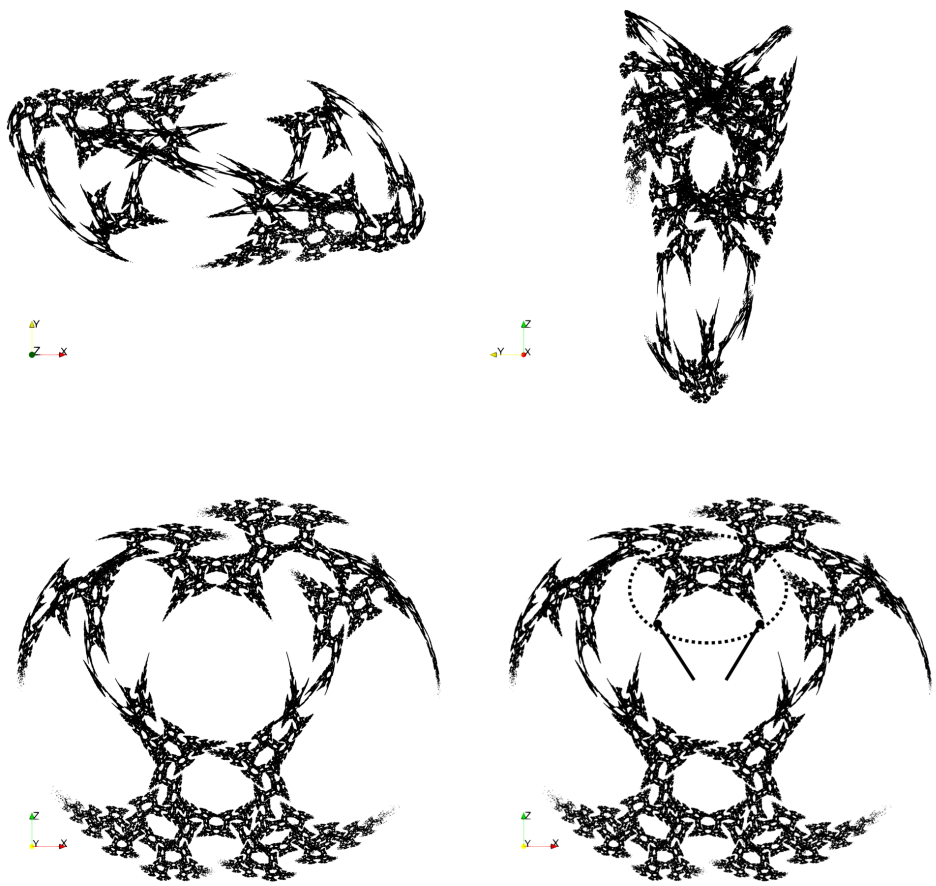

Figure 1: Hyperbolic triangle group $(3,3,4)$ with $I_2 I_3 I_1 I_3$ parabolic.

The visual clues are pointed out in the following examples. See figures 5, 6, 7 and 8.

IV — Redundancy

From the census of the unipotent boundary representations in [FKR15], we experimentally computed the limit sets. After a visual inspection, we kept the representations that gave fractals.

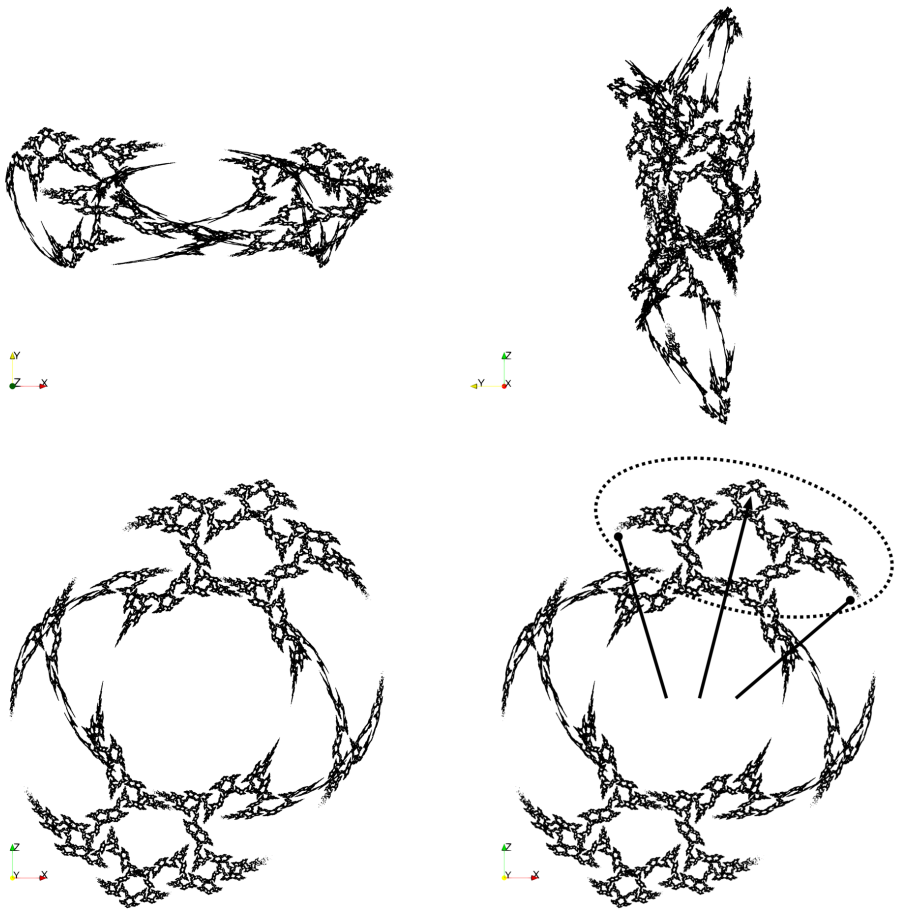

Figure 2: Hyperbolic triangle group $(3,3,5)$ with $I_2I_3I_1I_3$ parabolic.

Those representations come in pairs by complex conjugation of the coefficients. For each pair, we wrote down the identifier of one representation and of verified relations (they are true by exact computations) in the following table. Often, those relations allowed to reproduce the relation of the fundamental group. Each time, the fundamental group is presented by two generators and a relation.

In the next table (table 1), we present all the representations which gave a fractal. Those accompanied by a star won't be studied furthermore.

Additional remarks on the full experimental table It is already known that m004-1 and m004-3 are related by the composition of a figure-eight knot's symmetry (see [Fal08]). It might be the same for m045-1 and m045-8 but this remains to be proved.

The representation m038-1 presents the characteristics of a $(3,4,4)$ complex hyperbolic triangle group. Indeed, m038 has such a representation according to a preprint of Ma and Xie that the author has been able to consult. The representation m137-5 presents the characteristics of a $(3,4,5)$ complex hyperbolic triangle group.

The representations we selected all present characteristics of $(3,3,n)$ complex hyperbolic groups. We will indeed show this phenomenon.

Theorem IV.1 *For each column of table 2, the manifolds have a $(3,3,n)$ complex hyperbolic triangle group representation, with the normal subgroup of the even-length words for image. Furthermore, all those representations (for a shared a column) are the same, up to conjugation and complex conjugation.*

	id	$a^n = e$	$b^n = e$	$(a,b)^n = e$
m004-1	[0,0,0]	a^4	b^3	$(Ab)^3$
m004-3*	[0,1,0]			
m009-1	[0,1,2]	a^5		$(aaB)^3, (aab)^3$
m015-2	[0,1,2]	a^3	b^5	$(abb)^3$
m022-1	[0,0,0]	a^3	b^4	$(ab)^3$
m023-1	[0,0,0]		b^6	$(Abb)^3$, $(aB^3)^3$
m023-7*	[0,4,0]			
m029-1	[0,0,0]	a^3	b^4	$(aB)^3$
m032-7	[0,2,4]	a^3	b^7	$(ABB)^3$
m034-1	[0,0,0]	a^4	b^3	$(AB)^3$
m035-1*	[0,0,0]			$(Ba)^2$, $(aBAb^3)^2$
m038-1*	[0,0,0]	a^4	b^4	$(AB)^3$
m045-1*	[0,0,0]			$(aab)^2$, $(a^3bbaa)^2$
m045-8	[0,4,2]	a^7	b^3	$(aaaB)^3$
m053-1*	[0,0,0]			
m053-4*	[0,1,1]			
m053-7*	[0,2,2]			
m081-1	[0,0,0]	a^3	b^4	$(aB)^3$
m117-1	[0,0,0]	a^3	b^3	$(AB)^4$
m129-1	[0,0,0]	a^3	b^3	
m130-1*	[0,1,0]			$(ab)^2$, $(a^2b^3)^4$, $(a^2b^4)^2$
m137-5*	[0,2,2]	a^4	b^5	$(Abb)^3$
m142-1	[0,0,2]	a^3	b^3	$(aB)^5$
m146-3	[0,3,2]	a^3	b^5	$(AB)^3$
m203-1	[0,0,0]	a^3	b^3	

Table 1: Full experimental table.

$\Delta(3,3,4)$	$\Delta(3,3,5)$	$\Delta(3,3,6)$	$\Delta(3,3,7)$	$\Delta(3,3,\infty)$
m004*	m009*	m023*	(m039)*	m129*
m022	m015		m032	m203
m029	m142		m045	
m034	m146			
m081				
m117				

Table 2: Manifolds with $(3,3,n)$ complex hyperbolic triangle group representations.

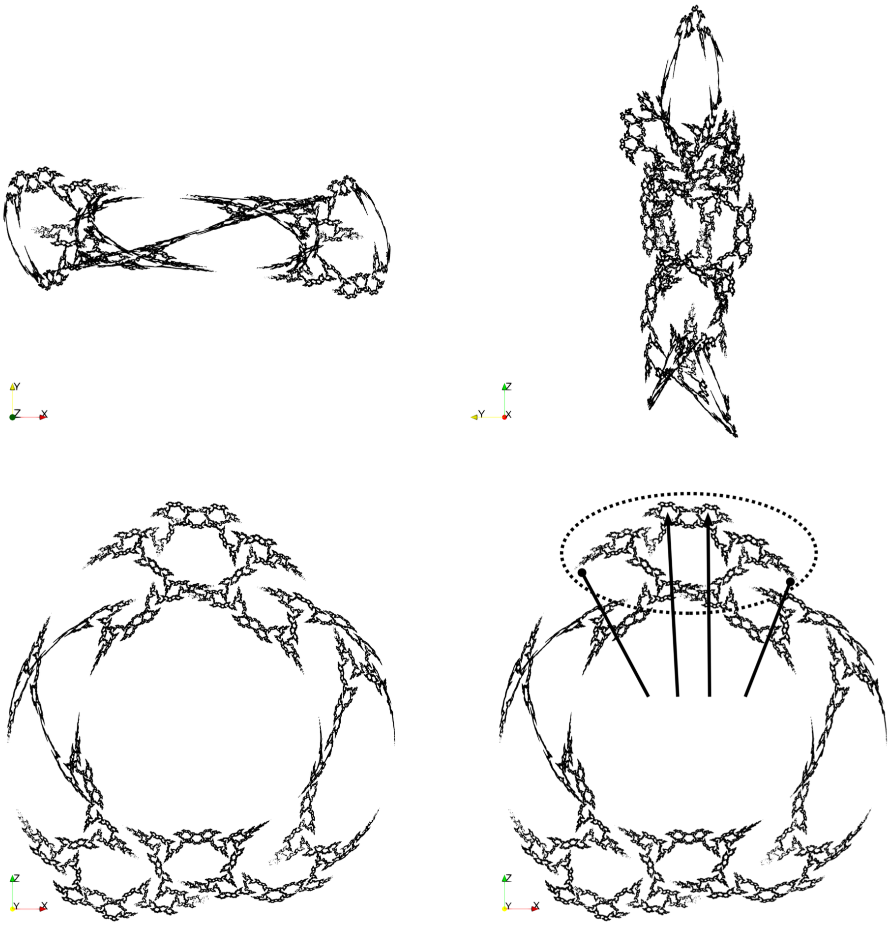

Figure 3: Hyperbolic triangle group $(3,3,6)$ with $I_2I_3I_1I_3$ parabolic.

By consequence, for each column, at most one representation is a uniformization of the corresponding manifold. Deraux [Der15] encountered the same phenomenon with the manifolds m009 and m015. In fact, we can complete the picture with the following result of Acosta.

Theorem IV.2 ([Aco19]) *Let $4 \leq n \leq \infty$. Let Γ be a hyperbolic $(3,3,n)$ triangle group. Suppose that $I_2I_3I_1I_3$ is parabolic unipotent. Let $\Gamma' \subset \Gamma$ be the subgroup of even-length words. Then the manifold at infinity of $\mathbf{H}^2_{\mathbf{C}}/\Gamma'$ is the Dehn surgery with slope $(1, n-3)$ on any cusp of the Whitehead link complement.*

With SnapPy, it is possible to compute Dehn surgeries on the Whitehead link complement. Remember that m129 is the Whitehead link complement. In each column, we can identify a uniforming representation (which must be unique in the column). (In order to make SnapPy work correctly, one needs to call the manifold 5^2_1 and fill a cusp with the meridian equal to $n-3$ and the longitude equal to 1.)

The manifolds in table 2 for which the corresponding $(3,3,n)$ representation is a uniformization were marked with a star. Those manifolds are: m004, m009, m023, m039 and m129. The first uniformization of m129 (the Whitehead link complement) was shown by Schwartz [Sch01], but the present uniformization by a $(3,3,\infty)$ triangle group with unipotent boundary was studied by Parker and Will [PW17].

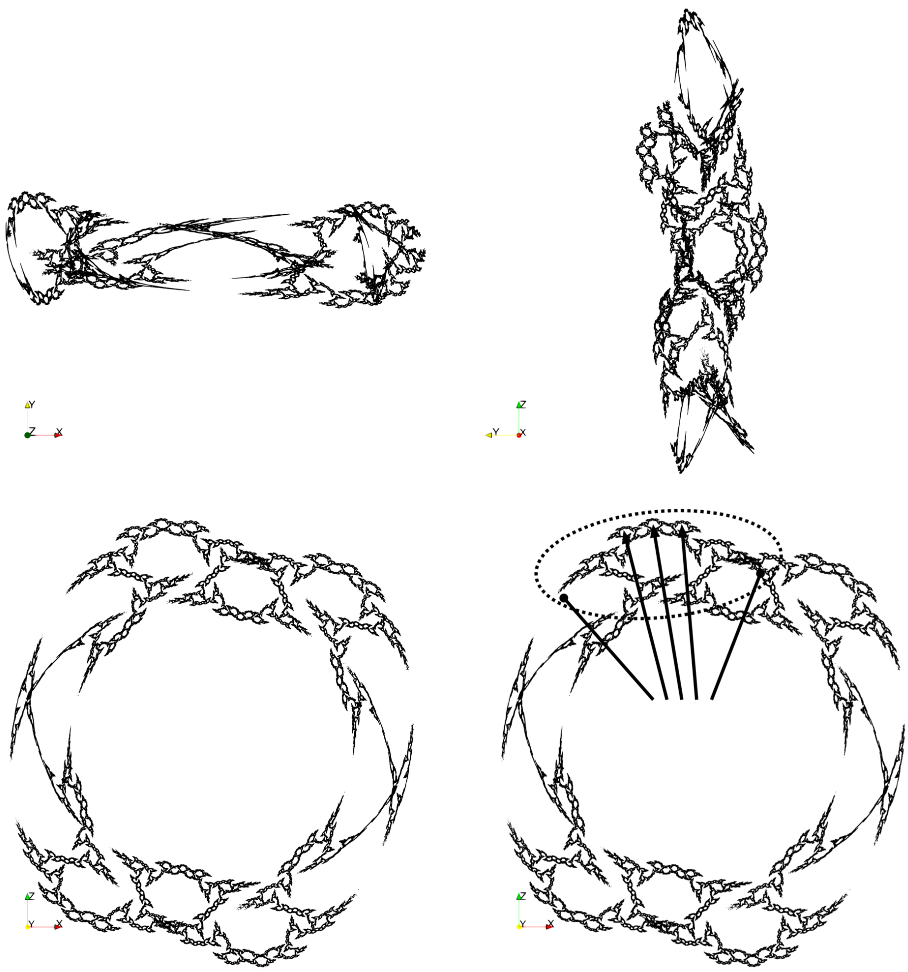

Figure 4: Hyperbolic triangle group $(3,3,7)$ with $I_2 I_3 I_1 I_3$ parabolic.

About m039 This manifold is described with 5 tetrahedra and does therefore not appear in the explicit census of [FKR15]. By the theorem of Acosta, it does have a representation with the even-length subgroup of the $(3,3,7)$ complex hyperbolic triangle group having $I_2 I_3 I_1 I_3$ parabolic for image. We will construct this representation and show that it has in fact parabolic unipotent boundary.

The same phenomenon happens for $s000$, the manifold obtained by Dehn surgery on the complement of the Whitehead link with the slope associated to the triangle group $(3,3,8)$.

The method In the following subsections, corresponding to the different values of n, we recognize the selected representations and reconstruct the triangle group representation in order to show the result with certainty. This is organized in three steps and for each one we give a table.

(1) The first table gives the informations from the census. That is to say, the way to retrieve the representation is the census (its identifier) and some relations (formally) verified that suggests the choice of the triangle group. Those relations always imply the fundamental group relation.

(2) The second table gives a morphism from the fundamental group of the variety to the abstract triangle group $\Lambda(3,3,n)$. This is achieved by giving a presentation of the fundamental group (that is always constituted of two generators and a relation) and the specification of the generators' images. We check that

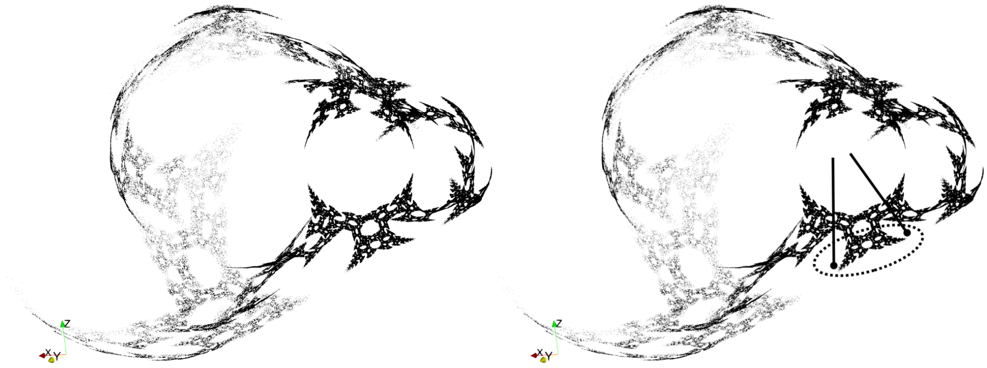

Figure 5: The representation m022-1 is of type $(3,3,4)$.

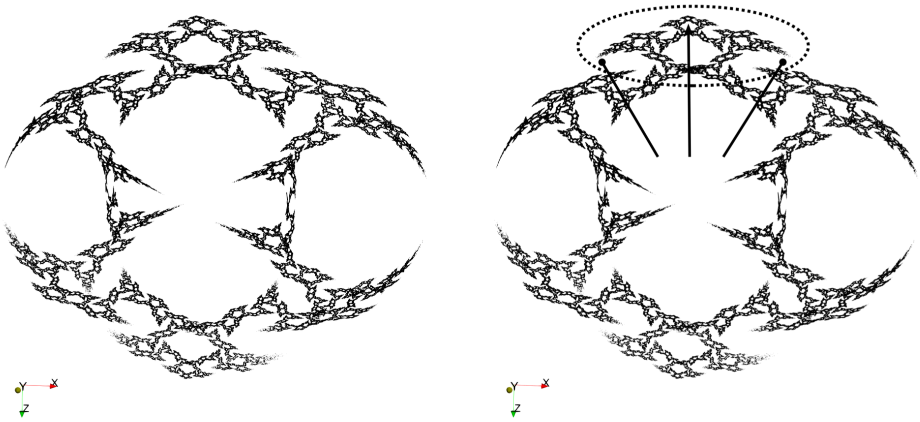

Figure 6: The representation m015-2 is of type $(3,3,5)$.

the morphism chosen verifies the relations given in the preceding table and (therefore) the fundamental group relation (showing that it is indeed a morphism). In this table, we also precise what the word corresponding to $I_2 I_3 I_1 I_3$ is. This word can be experimentally computed with the representation of the census, and we can check that it is indeed parabolic unipotent (for one can take the trace once it is verified that the matrix is in $\mathrm{SU}(2,1)$).

(3) In the third table, we compute the peripheral holonomy. Equivalences of words are given according to the relations inscribed in the first table (that are verified by the abstract morphism previously constructed). This peripheral holonomy is computed in terms of $I_2 I_3 I_1 I_3$. This implies that the abstract representation, once embedded in $\mathrm{PU}(2,1)$ with the convenient angular invariant (that makes $I_2 I_3 I_1 I_3$ parabolic unipotent) must exist in the census. Therefore, this constructed representation is indeed conjugated (or complex conjugated) to the one selected in the census, by unicity of the complex hyperbolic triangle group representations that has $I_2 I_3 I_1 I_3$ parabolic.

IV.1 $(3,3,4)$ – m004, m022, m029, m034, m081 and m117

The first table is computed with the help of the following code.

Figure 7: The representation m023-1 is of type $(3,3,6)$.

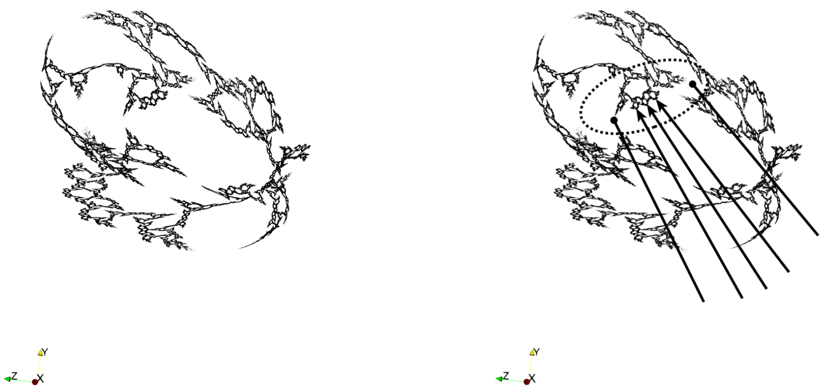

Figure 8: The representation m032-7 is of type $(3,3,7)$.

```
import snappy
import numpy

# Data
s = 'm022'
i,j,k = 0,0,0
#

M = snappy.Manifold(s)
G = M.fundamental_group()
P = M.ptolemy_variety(3,'all').retrieve_solutions(prefer_rur=True,
                                  verbose=False)
S = [[component
      for component in per_obstruction
      if component.dimension == 0]
     for per_obstruction in P]
K = S[i][j]

def f(x):
    mat_x = K.evaluate_word(x,G)
    return [[z.lift() for z in y] for y in mat_x]

# Search trivial word
```

```python
word = 'a' # to be changed manually
for i in range(1,50):
    s = f(word*i)
    if s == [[1, 0, 0], [0, 1, 0], [0, 0, 1]]:
        print(i,s)
        break
```

	id	$a^n = e$	$b^n = e$	$(a,b)^n = e$
m004-1	[0,0,0]	a^4	b^3	$(Ab)^3$
m022-1	[0,0,0]	a^3	b^4	$(ab)^3$
m029-1	[0,0,0]	a^3	b^4	$(aB)^3$
m034-1	[0,0,0]	a^4	b^3	$(AB)^3$
m081-1	[0,0,0]	a^3	b^4	$(aB)^3$
m117-1	[0,0,0]	a^3	b^3	$(AB)^4$

Here, we consider the abstract triangle group

$$\Lambda(3,3,4) = \left\langle I_1, I_2, I_3\,; \quad \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1I_2)^3 = (I_2I_3)^3 = (I_3I_1)^4 = e \end{array} \right\rangle . \tag{25}$$

	a	b	fundamental group's relation	$I_2I_3I_1I_3 = P$
m004	I_3I_1	I_3I_2	aabABBAba	BA
m022	I_3I_2	I_1I_3	abbbbbabAAb	Ab
m029	I_2I_1	I_3I_1	aBabbbAAbbb	aBB
m034	I_3I_1	I_1I_2	aaabbABAbb	BAA
m081	I_2I_1	I_3I_1	abbbaBaaaaB	aBB
m117	I_3I_2	$I_3I_1I_2I_3$	aabbaabbABAbb	aB

	peripheral curves	
m004	ab P^{-1}	$aBAbABab \equiv (ab)^3$ P^{-3}
m022	Ba P^{-1}	$AAbabA \equiv Ba$ P^{-1}
m029	$abb \equiv aBB$ P	$bAAAbbb \equiv e$ e
m034	$bbaa \equiv BAA$ P	$AAABBBA \equiv e$ e
m081	$bba \equiv BBa$ $I_1P^{-1}I_1$	$BaaaBa \equiv BBa$ $I_1P^{-1}I_1$
m117	bA P^{-1}	$BAAABA \equiv bA$ P^{-1}

IV.2 (3,3,5) – m009, m015, m142 and m146

We iterate the same process.

	id	$a^n = e$	$b^n = e$	$(a,b)^n = e$
m009-1	[0,1,2]	a^5		$(aaB)^3, (aab)^3$
m015-2	[0,1,2]	a^3	b^5	$(abb)^3$
m142-1	[0,0,2]	a^3	b^3	$(aB)^5$
m146-3	[0,3,2]	a^3	b^5	$(AB)^3$

This time, the triangle group is

$$\Lambda(3,3,5) = \left\langle I_1, I_2, I_3\,;\quad \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1 I_2)^3 = (I_2 I_3)^3 = (I_3 I_1)^5 = e \end{array} \right\rangle. \tag{26}$$

	a	b	fundamental group's relation	$I_2 I_3 I_1 I_3 = P$
m009	$I_1 I_3$	$I_3 I_1 I_3 I_1 I_3 I_2$	aabABaaBAb	BA
m015	$I_2 I_1$	$I_3 I_1 I_3 I_1$	abbAAbbaBBB	aB
m142	$I_3 I_1 I_2 I_3$	$I_2 I_3$	abbaBabbaBaaaaB	BA
m146	$I_2 I_3$	$I_3 I_1$	aabbaaabbaaBAB	aB

	peripheral curves	
m009	ab P^{-1}	$ABaaaBAb \equiv (ab)^2$ P^{-1}
m015	bA P^{-1}	$abbAAAbb \equiv (bA)^{-1}$ P
m142	BA P	$bAAAbA \equiv BA$ P
m146	$baa \equiv bA$ P^{-1}	$BABABB \equiv aB$ P

IV.3 $(3,3,6)$ – m023

	id	$a^n = e$	$b^n = e$	$(a,b)^n = e$
m023-1	[0,0,0]		b^6	$(Abb)^3, (aB^3)^3$

The triangle group is

$$\Delta(3,3,6) = \left\langle I_1, I_2, I_3\,;\quad \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1 I_2)^3 = (I_2 I_3)^3 = (I_3 I_1)^6 = e \end{array} \right\rangle. \tag{27}$$

	a	b	fundamental group's relation	$I_2 I_3 I_1 I_3 = P$
m023	$I_3 I_1 I_3 I_1 I_3 I_2 I_3 I_1$	$I_1 I_3$	aBAbbABabbb	BAB

	peripheral curves	
m023	bab P^{-1}	$bbaBABabb \equiv (BAB)^2$ P^2

IV.4 $(3,3,7)$ – m032, m045 and m039

	id	$a^n = e$	$b^n = e$	$(a,b)^n = e$
m032-7	[0,2,4]	a^3	b^7	$(ABB)^3$
m045-8	[0,4,2]	a^7	b^3	$(aaaB)^3$

The triangle group is

$$\Lambda(3,3,7) = \left\langle I_1, I_2, I_3\,;\quad \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1 I_2)^3 = (I_2 I_3)^3 = (I_3 I_1)^7 = e \end{array} \right\rangle. \tag{28}$$

	a	b	fundamental group's relation	$I_2 I_3 I_1 I_3 = P$
m032	$I_1 I_2$	$I_1 I_3 I_1 I_3 I_1 I_3$	aaBBAbbbbbABB	$Abbb$
m045	$I_1 I_3 I_1 I_3$	$I_2 I_1$	aaabbaaaBAAAAB	ba

	peripheral curves	
m032	$BBBa$	$bbAAAbba \equiv BBBa$
	P^{-1}	P^{-1}
m045	AB	$bAAABBBAAA \equiv ba$
	P^{-1}	P

We additionally construct a representation for m039 so that we complete our picture.

	a	b	fundamental group's relation
m039	I_1I_3	$I_3I_1I_3I_1I_2I_1$	aabABaaaaBAb

This representation verifies $a^7 = (aab)^3 = (Baaaa)^3$ and this implies the fundamental group relation.

The peripheral holonomy is prescribed by $U = AB$ and $V = abABAbaaba$. The images are respectively $\rho(U) = 321313$, $\rho(V) = 312132312123$ (where we denoted $1,2,3$ instead of I_1, I_2, I_3). We can check that $\rho(U) = \rho(V)$ by computing $\rho(U)^{-1}\rho(V)$.

Once θ is fixed in order to have a triangle representation with $I_2I_3I_1I_3$ parabolic unipotent, $\rho(V)$ has its trace equal to 3 and this representation therefore has parabolic unipotent boundary.

IV.5 $(3,3,\infty)$ – m129 and m203

	id	$a^n = e$	$b^n = e$	$(a,b)^n = e$
m129-1	[0,0,0]	a^3	b^3	
m203-1	[0,0,0]	a^3	b^3	

The triangle group is

$$\Lambda(3,3,\infty) = \left\langle I_1, I_2, I_3\,; \quad \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1I_2)^3 = (I_2I_3)^3 = e \end{array} \right\rangle. \tag{29}$$

This time, an additional parabolic unipotent element is given by I_1I_3.

	a	b	fundamental group's relation	$I_2I_3I_1I_3 = P$
m129	$I_3I_1I_2I_3$	I_2I_3	aaaBBabAAAbbAB	BA
m203	$I_3I_1I_2I_3$	I_2I_3	aaabbaaBAAABBAAb	BA

	peripheral curves	
m129	$AAb \equiv ab$	$AAAbbA \equiv BA$
	P^{-1}	P
	Ab	$bAAAba \equiv Ba$
	$(I_3I_2)I_1I_3(I_3I_2)^{-1}$	$(I_3I_2)(I_1I_3)^{-1}(I_3I_2)^{-1}$
m203	$aab \equiv Ab$	$BBAAAB \equiv e$
	$(I_3I_2)I_1I_3(I_3I_2)^{-1}$	e
	ab	$BAAABBAAA \equiv e$
	P^{-1}	e

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